Signals and Systems - MAE 143A

Final Exam - Winter Quarter 2008

Student name and number __________________________________

For all the questions you need to show ALL your work to get to the answer.

1. (7 points) Consider the system

$$\begin{cases} \dot{x}_1 = x_2 + u\cos x_2 \\ \dot{x}_2 = \dfrac{x_1^3}{4} - x_1 + x_2 \end{cases}$$

where u is the system input and (x_1, x_2) is the system output.

(i) What are the equilibrium points of the system?

(ii) Linearize the system about $(x_1, x_2) = (2, 0)$ and $u = 0$.

Solution: (i) The equilibrium points are the solutions to the equations:

$$0 = x_2\,, \qquad 0 = \frac{x_1^3}{4} - x_1 + x_2\,.$$

From the first equation $x_2 = 0$ and, substituting this into the second equation:

$$0 = \frac{x_1^3}{4} - x_1 \qquad \Longleftrightarrow \qquad x_1^2 = 4 \text{ or } x_1 = 0\,, \qquad \Longleftrightarrow \qquad x_1 = \pm 2, \text{ or } x_1 = 0\,.$$

The set of three equilibrium points is then $\{(0,0), (2,0), (-2,0)\}$.

(ii) To linearize the system about $(2, 0)$ we follow the steps in the linearization method. First, we rewrite the system in vector form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + u\cos x_2 \\ \dfrac{x_1^3}{4} - x_1 + x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = f(x_1, x_2)\,.$$

Next, we determine the Jacobian of f and value it at $(x_1, x_2) = (2, 0)$ and $u = 0$:

$$\frac{\partial f}{\partial x}\Big|_{((2,0),0)} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}_{|((2,0),0)} = \begin{bmatrix} 0 & 1 \\ \dfrac{3x_1^4}{4} - 1 & 1 \end{bmatrix}_{|((2,0),0)} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

We determine the derivative of f with respect to u and value it at $(x_1, x_2) = (2, 0)$ and $u = 0$:

$$\frac{\partial f}{\partial u}\Big|_{((2,0),0)} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\Big|_{((2,0),0)} = \begin{bmatrix} \cos\, x_2 \\ 0 \end{bmatrix}\Big|_{((2,0),0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In this way, we get a linear system in the new variables $(y_1, y_2) = (x_1 - 2, x_2)$ as follows:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \, .$$

2. (5 points) What is the expression of the signal $x_2(t)$ such that:

$$\mathcal{L}\{\mathrm{e}^{-t+2} 1(t-5) * x_2(t)\} = \frac{\mathrm{e}^{-5s-3}}{(s+1)(s^2+1)} \, ?$$

Explain what properties of the Laplace transform you use to get to the result.

Solution: We use the properties of the Laplace transform to determine $x_2(t)$. By the convolution-multiplication duality, we have that:

$$\mathcal{L}\{\mathrm{e}^{-t+2} 1(t-5) * x_2(t)\} = \mathcal{L}\{\mathrm{e}^{-t+2} 1(t-5)\}\mathcal{L}\{x_2(t)\} = \frac{\mathrm{e}^{-5s-3}}{(s+1)(s^2+1)} \, . \tag{1}$$

Let us compute now $\mathcal{L}\{\mathrm{e}^{-t+2} 1(t-5)\}$. Adjusting the constant in the exponential and by the property of linearity of the Laplace transform, we have that:

$$\mathcal{L}\{\mathrm{e}^{-t+2} 1(t-5)\} = \mathcal{L}\{\mathrm{e}^{-(t-5)-3} 1(t-5)\} = \mathrm{e}^{-3}\mathcal{L}\{\mathrm{e}^{-(t-5)} 1(t-5)\} \, .$$

By the time-shifting property of the Laplace transform, we have that:

$$\mathrm{e}^{-3}\mathcal{L}\{\mathrm{e}^{-(t-5)} 1(t-5)\} = \mathrm{e}^{-3}\mathrm{e}^{-5s}\mathcal{L}\{\mathrm{e}^{-t} 1(t)\} = \frac{\mathrm{e}^{-3-5s}}{s+1} \, .$$

Substituting this expression into (1), we obtain:

$$\frac{\mathrm{e}^{-3-5s}}{s+1}\mathcal{L}\{x_2(t)\} = \frac{\mathrm{e}^{-5s-3}}{(s+1)(s^2+1)} \quad \Longleftrightarrow \quad \mathcal{L}\{x_2(t)\} = \frac{s+1}{(s+1)(s^2+1)} = \frac{1}{s^2+1}$$

In this way,

$$x_2(t) = \mathcal{L}^{-1}\{\frac{1}{s^2+1}\} = \sin t \cdot 1(t) \, .$$

3. (6 points) Consider the ODE system:

$$2\dot{y}(t) + y(t) = \dot{u}(t) + 3u(t) \, .$$

What is its impulse response $h(t)$? Use the time-domain approach to determine $h(t)$, and do not use transfer functions or the Laplace transform method.

Solution: The time-domain approach to determine $h(t)$ requires that when the highest derivative of $y(t)$ in the ODE is equal to the highest derivative of $u(t)$ in the ODE, we must consider:

$$h(t) = y_h(t) \cdot 1(t) + K_\delta \delta(t)\,,$$

where $y_h(t)$ is the solution to the associated homogeneous equation.

In this case we have that

$$2\dot{y}(t) + y(t) = 0 \quad \implies \quad y_h(t) = K\mathrm{e}^{-t/2}\,.$$

To determine the impulse response $h(t) = K\mathrm{e}^{-t/2} \cdot 1(t) + K_\delta \delta(t)$, we must determine the constants K and K_δ. We do this by taking generalized derivatives of $h(t)$ and substituting them into the ODE with $u(t) = \delta(t)$. We have:

$$\dot{h}(t) = -\frac{K}{2}\mathrm{e}^{-t/2} \cdot 1(t) + K\mathrm{e}^{-t/2} \cdot \delta(t) + K_\delta \dot{\delta}(t)\,,$$

then,

$$-2\frac{K}{2}\mathrm{e}^{-t/2} \cdot 1(t) + 2K\mathrm{e}^{-t/2} \cdot \delta(t) + 2K_\delta \dot{\delta}(t) + K\mathrm{e}^{-t/2} \cdot 1(t) + K_\delta \delta(t) = \dot{\delta}(t) + 3\delta(t)\,.$$

This simplifies to:

$$2K\mathrm{e}^{-t/2} \cdot \delta(t) + 2K_\delta \dot{\delta}(t) + K_\delta \delta(t) = \dot{\delta}(t) + 3\delta(t)\,.$$

Equating similar terms in both sides of the equation leads to:

$$\begin{aligned} &2K_\delta \dot{\delta}(t) = \dot{\delta}(t) \quad \implies K_\delta = \frac{1}{2}, \\ &2K\mathrm{e}^{-t/2} \cdot \delta(t) + K_\delta \delta(t) = 3\delta(t) \qquad \implies \quad 2K + \frac{1}{2} = 3 \quad \implies K = \frac{5}{4}\,. \end{aligned}$$

In all, we have that $h(t) = \frac{5}{4}\mathrm{e}^{-t/2} \cdot 1(t) + \frac{1}{2}\delta(t)$.

4. (6 points) Suppose that the impulse response of a system is given by $h(t) = 3\delta(t-3) + 2 \cdot 1(t)$.

   What is the system response to a unit ramp? (Use the convolution integral and its properties to compute this response.)

Solution: The system response is

$$y(t) = h(t) \ast r(t) = (3\delta(t-3) + \mathrm{e}^{-2t} \cdot 1(t)) \ast r(t) = 3\delta(t-3) \ast r(t) + 2 \cdot 1(t) \ast r(t)\,.$$

We now compute each term of the sum independently. First,

$$y_1(t) = 3\delta(t-3) \ast r(t) = 3r(t-3) = 3(t-3) \cdot 1(t-3)\,,$$

Let us now compute $y_2(t) = 2 \cdot 1(t) \ast r(t)$. By the differentiation property of convolution, we have that:

$$\dot{y}_2(t) = 2 \cdot 1(t) \ast 1(t)\,, \text{ and } \ddot{y}_2(t) = 2 \cdot 1(t) \ast \delta(t) = 2 \cdot 1(t)\,.$$

Therefore,

$$\begin{aligned}
\dot{y}_2(t) &= \int_{-\infty}^{t} 2 \cdot 1(\tau)\, d\tau = 2\tau\Big|_0^t 1(t) = 2t \cdot 1(t)\,,\\
y_2(t) &= \int_{-\infty}^{t} 2\tau \cdot 1(\tau) d\tau = \left(\int_0^t 2\tau \cdot 1(\tau) d\tau\right) 1(t) = (\tau^2\Big|_0^t 1(t) = t^2 \cdot 1(t)\,.
\end{aligned}$$

In all, we have that

$$y(t) = \big(3(t-3) \cdot 1(t-3) + t^2\big)\, 1(t)\,.$$

5. (6 points) A system has a transfer function given by $H(s) = \dfrac{3s+1}{(s^2+3s+4)(s^2+2s)}$. Use the Final Value theorem to compute the final value of the impulse response. You have to justify why you can use this result.

Solution: The impulse response is $h(t)$ such that $\mathcal{L}\{h(t)\} = H(s)$. To compute the final value of $h(t)$ we can try to use the Final Value theorem.

To be able to apply the theorem, we need to guarantee that the poles of $sH(s)$ are in the open left half plane. In this case,

$$sH(s) = \frac{3s+1}{(s^2+3s+4)(s+2)}$$

with poles $p_1 = -2$ and $p_{2,3} = \frac{-3 \pm \sqrt{9-16}}{2}$. These poles satisfy that $Re(p_i) < 0$ for $i \in \{1, 2, 3\}$. Therefore, we can apply the Final Value theorem and obtain that:

$$\lim_{t \to +\infty} h(t) = \lim_{s \to 0} sH(s) = \lim_{s \to 0} \frac{3s+1}{(s^2+3s+4)(s+2)} = \frac{1}{8}\,.$$

6. (10 points) Suppose a system with transfer function $H_1(s) = \dfrac{1}{s^2+2s-3}$ is connected in series with another with trasfer function $H_2(s) = \dfrac{1}{s+3}$. What is the interconnected system impulse

response? What is its transfer function? Is the system stable?

Solution: The cascaded system has a transfer function which is the product of the two transfer functions given. That is:

$$H(s) = H_1(s)H_2(s) = \frac{1}{(s^2+2s-3)(s+3)}\,.$$

Now the impulse response can be obtained as $h(s) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2-2s-3)(s+3)}\right\}$. Observe that:

$$s^2+2s-3 = (s-1)(s+3)\,.$$

In order to find the expression of $h(t)$, we first obtain the partial fraction expansion of $\frac{1}{(s-1)(s+3)^2}$. Since there are two repeated poles, the expansion is of the form:

$$\frac{1}{(s-1)(s+3)^2} = \frac{K_{11}}{s+3} + \frac{K_{12}}{(s+3)^2} + \frac{K_2}{s-1}\,.$$

The constants are determined by the following formulas:

$$\begin{aligned}
K_{12} &= \frac{(s+3)^2}{(s-1)(s+3)^2}\Big|_{(s=-3)} = \frac{1}{(s-1)}\Big|_{(s=-3)} = \frac{1}{-4}\,,\\
K_{11} &= \frac{d}{ds}\left(\frac{(s+3)^2}{(s-1)(s+3)^2}\right)\Big|_{(s=-3)} = \frac{d}{ds}\left(\frac{1}{s-1}\right)\Big|_{(s=-3)} = \frac{-1}{(s-1)^2}\Big|_{(s=-3)} = \frac{-1}{16}\,,\\
K_2 &= \frac{s-1}{(s-1)(s+3)^2}\Big|_{(s=1)} = \frac{1}{16}\,.
\end{aligned}$$

Therefore, we have that:

$$h(t) = \mathcal{L}^{-1}\left\{\frac{-1}{4}\frac{1}{(s+3)^2} + \frac{-1}{16}\frac{1}{s+3} + \frac{1}{4}\frac{1}{s-1}\right\} = -\frac{1}{4}t\mathrm{e}^{-3t}\cdot 1(t) - \frac{1}{16}\mathrm{e}^{-3t}\cdot 1(t) + \frac{1}{16}\mathrm{e}^{t}\cdot 1(t)\,.$$

The system is unstable because there is a pole $p = 1$ with a positive real part.

7. (10 points) Consider the discrete-time system

$$y[n] + \frac{j}{2}y[n-1] = u[n] - 2u[n-1] + u[n-2]\,.$$

What is the discrete-time impulse response? (Use the time-domain method to compute it.) Is the system stable? Compute the system transfer function by taking the $\mathcal{Z}$-transform of the impulse response.

Solution: To compute the impulse response we first compute the solution to:

$$h_1[n] + \frac{j}{2}h_1[n-1] = \delta[n]\,. \tag{2}$$

In this case, the signal $h_1[n]$ will have the form $h_1[n] = K_1\alpha_1^n \cdot 1[n]$ with α_1 the root of

$$\alpha_1^n + \frac{j}{2}\alpha_1^{n-1} = 0\,.$$

Setting $n = 1$ we get $\alpha_1 + \frac{j}{2} = 0$. Therefore $\alpha_1 = -\frac{j}{2}$. Now, to compute the value of the constant K_1, we substitute $h_1[n]$ in (2) and give values to n. We have that:

$$h_1[0] + \frac{j}{2}h_1[-1] = \delta[0] = 1 \qquad \Longrightarrow \qquad h_1[0] = 1\,.$$

On the other hand, $h_1[0] = K_1(-\frac{j}{2})^0 \cdot 1[0] = 1$. In all, we have that $h_1[n] = (-\frac{j}{2})^n \cdot 1[n]$. Now, the overall system response is obtained by time-shifting and amplifying $h_1[n]$. We have that:

$$\begin{aligned} h[n] &= h_1[n] - 2h_1[n-1] + h_1[n-2] \\ &= (-\frac{j}{2})^n \cdot 1[n] - 2(-\frac{j}{2})^{n-1} \cdot 1[n-1] + (-\frac{j}{2})^{n-2} \cdot 1[n-2]\,. \end{aligned}$$

The system is stable because the response to the input δ is summable:

$$\sum_{n=0}^{+\infty} |h[n]| < +\infty\,.$$

This is true because we are summing exponentials depending on $\alpha_1 = -\frac{j}{2}$ with modulus $|-\frac{j}{2}| = \frac{1}{2} < 1$ strictly less than one.

The DT system transfer function computed from the impulse response is:

$$\mathcal{Z}\{h[n]\} = \frac{z}{z+\frac{j}{2}} - 2z^{-1}\frac{z}{z+\frac{j}{2}} + \frac{z^{-2}z}{z+\frac{j}{2}} = \frac{z-2+z^{-1}}{z+\frac{j}{2}}$$