Practice Questions for Quiz 2, 2009

The material for Quiz 2 will be based on the Homework (2 and 3), the slides (systemsCT.pdf and ImpulseResponse.pdf) and the text book (this is material included in chapters 4 and 6 which is also common to the slides.) The following are some additional examples for you to practice before Quiz 2. The real quiz questions can either be similar to these exercises or to other exercises like the ones in Homework 2 and 3. Because of this, please study and practice all the homework problems as well.

Before you start: Recall that for all the questions in the Quiz or exams you need to show ALL your work to get to the answer; otherwise you may lose some marks. When doing the real quiz, you should read the questions very carefully and, in case of doubt, ask a TA or the instructor for clarification.

I. Questions on System Models. The objective of these problems is to find system models based on ODEs or state-space representations. You should practice on how to find models of simple translational/rotational mechanical systems and of basic circuits.

1. (5 points) Find ODE models for the mechanical system shown in Figure 1.

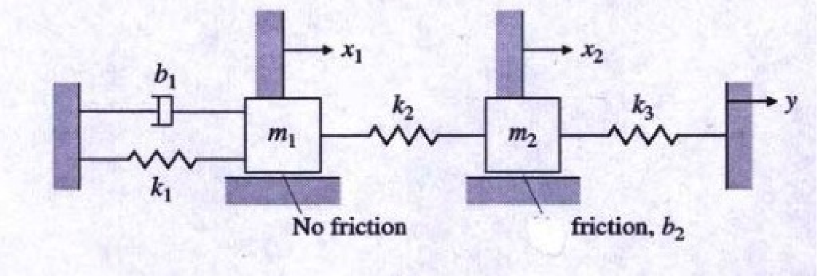

Figure 1: Multibody system for Problem 1

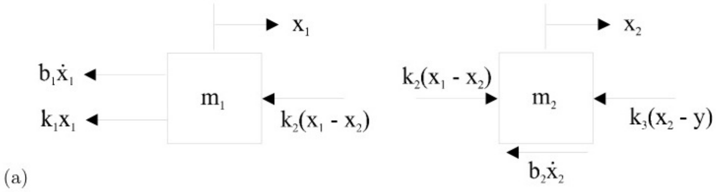

Figure 2: Multibody diagram for Problem 1

Solution: The key is to draw the free body diagram in order to keep the signs right; see Figure 2. To identify the direction of the spring forces on the object, let $x_2 = 0$ and fixe and increase x_1 from 0. Then the k_1 spring will be stretched producing its spring force to the left also. You can use the same technique on the damper forces and the other mass.

$$m_1\ddot{x}_1 = -k_1x_1 - b_1\dot{x}_1 - k_2(x_1 - x_2)$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - k_3(x_2 - y) - b_2\dot{x}_2$$

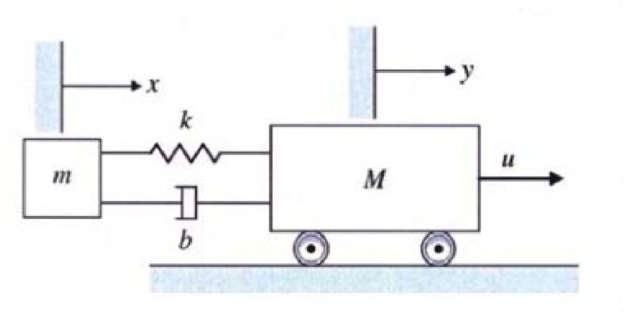

Figure 3: Multibody system for Problem 2

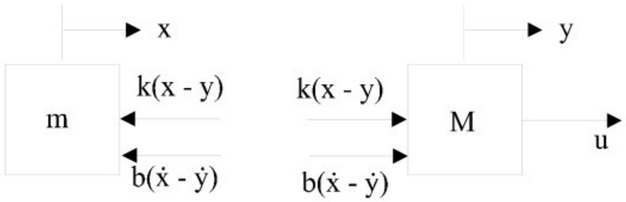

Figure 4: Multibody diagram for Problem 2

2. (5 points) In many mechanical positioning systems there is flexibility between one part of the system and another. Figure 4 depicts such a situation, where a force U is applied to the mass M and another mass m is connected to it. The coupling between the objects if often modeled by a spring constant k with a damping coefficient b, although the actual situation is usually much more complicated than this. Write the equations of motion governing this system.

Solution: The diagram of forces applied to this system is the following:

This results in the equations:

$$\begin{aligned} m\ddot{x} &= -k(x-y) - b(\dot{x}-\dot{y}) \\ M\ddot{y} &= U + k(x-y) + b(\dot{x}-\dot{y}) \end{aligned}$$

or, also

$$\begin{aligned} \ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} - \frac{k}{m}y - \frac{b}{m}\dot{y} &= 0 \\ -\frac{k}{M}x - \frac{b}{M}\dot{x} + \ddot{y} + \frac{k}{M}y + \frac{b}{M}\dot{y} &= \frac{1}{M}U\,. \end{aligned}$$

3. (6 points) Find a model of the circuit in Figure 5 as a system of ODEs. (The ODEs will contain first-order derivatives of the current going through the inductor, i_L, and the output voltage, V.) The input voltage is U and the output voltage is V.

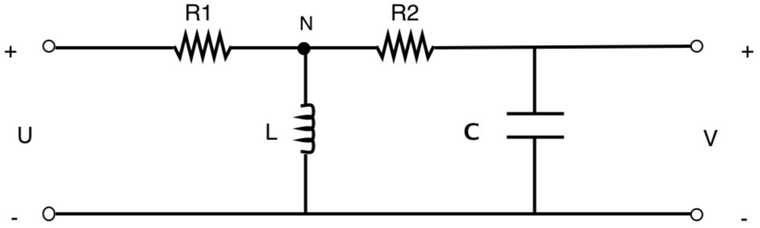

Figure 5: Circuit diagram for Problem 3

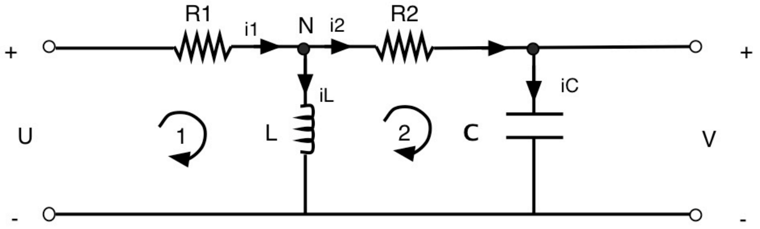

Figure 6: Worked out diagram for Problem 3

Solution: Consider the worked diagram in Figure 6. Let i_1 be the current passing through R_1, i_2 the current passing through R_2, i_L the current passing through L and i_C the current passing through C, respectively.

Applying KCL at N, we have:

$$i_1 = i_2 + i_L \tag{1}$$

By the KVL at loop 1:

$$U - i_1 R_1 - L\frac{di_L}{dt} = 0\,, \tag{2}$$

and the KVL at loop 2:

$$-\, i_2 R_2 - V + L\frac{di_L}{dt} = 0\,. \tag{3}$$

Using the element equation at the capacitor we get:

$$C\frac{dV}{dt} = i_2 = i_C\,. \tag{4}$$

We have four independent equations and four unknows V, i_1, i_2, i_L and therefore a closed system we can solve. We can simplify now the equations as follows. Using (9) and (12) we have

$$i_1 = C\frac{dV}{dt} + i_L\,. \tag{5}$$

Combining equation (5) and (10) leads to:

$$U - R_1C\frac{dV}{dt} - i_LR_1 - L\frac{di_L}{dt} = 0\,, \tag{6}$$

and the combination of equation (11) and (12),

$$-CR_2\frac{dV}{dt} - V + L\frac{di_L}{dt} = 0\,. \tag{7}$$

Equations (6) and (7) are a closed system with two unknowns i_L and V. In other words, we can describe the circuit by the system of equations:

$$\begin{aligned} L\frac{di_L}{dt} + R_1C\frac{dV}{dt} &= -i_LR_1 + U\,,\\ L\frac{di_L}{dt} - CR_2\frac{dV}{dt} - V &= 0\,. \end{aligned}$$

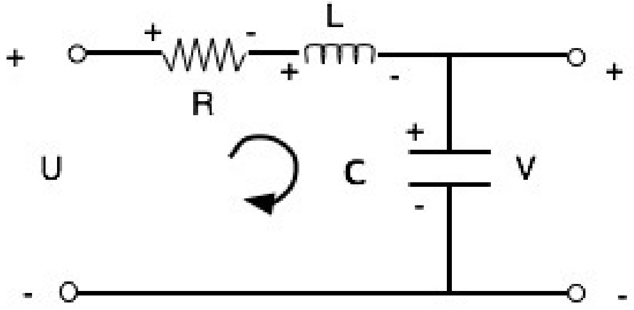

Figure 7: Circuit diagram for Problem 4

4. (10 points) The circuit in Figure 7 represents a “practical passive filter”. The circuit is excited a voltage difference $U(t)$ and has a response of $V(t)$. Determine:
   (i) A model of the circuit in the form of a second-order ODE. *Hint:* Take derivatives with respect to time in the capacitor law and combine it with the inductor law.
   (ii) The state-space representation of the system. Write the state-space model in matrix form.

Solution:

(i) An ODE model for this system can be found using the the KVL and circuit element laws. Let i be the current going through the resistor, inductor and capacitor. Let V_R, V_L and V_C be the voltage differences across R, L, and C, respectively.

By the KCL, we have that $i_R = i_C = i_L$.

By the KVL, we have that:

$$U - V_R - V_L - V_C = 0\,, \qquad \text{using KVL for the marked loop}\,, \tag{8}$$

$$V_C - V = 0\,, \qquad \text{using also KVL for the loop with } C \text{ and } V \tag{9}$$

The circuit element laws are given by:

$$V_R = iR\,, \tag{10}$$

$$L\frac{di}{dt} = V_L \quad \Longleftrightarrow \quad \frac{di}{dt} = \frac{1}{L}V_L \tag{11}$$

$$i = C\frac{dV_C}{dt}\,, \tag{12}$$

By equation (9), we have that $V_C = V$. Since everything has to be expressed in terms of V, we differentiate with respect to time in equation (12). Putting this together with equation (11), we obtain V_L in terms of the second derivative of V:

$$\begin{cases} \dfrac{di}{dt} = C\dfrac{d^2V}{dt^2} \\ \\ \dfrac{di}{dt} = \dfrac{1}{L}V_L \end{cases} \qquad \Longrightarrow \qquad V_L = LC\,\frac{d^2V}{dt^2}$$

On the other hand, (10) and (12) give that $V_R = CR\dfrac{dV}{dt}$. Now, using equation (8), and substituting the obtained values of V_L and V_R, we have that:

$$U = CR\frac{dV}{dt} + LC\frac{d^2V}{dt^2} + V \tag{13}$$

(ii) There is only one unknown in the system and that is the response V. Since the ODE is second order, the state dimension will be 2. Let us denote by $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ the state vector variable. We have that:

$$x_1 = V\,, \qquad x_2 = \dot{V}\,.$$

Then, $\dot{x}_1 = x_2$ and, using equation (13), $\dot{x}_2 = -\dfrac{1}{LC}V - \dfrac{CR}{LC}\dot{V} - \dfrac{1}{LC}U$. From here we have:

$$\dot{x}_1 = x_2\,,$$

$$\dot{x}_2 = -\frac{1}{LC}x_1 - \frac{R}{L}x_2 - \frac{1}{LC}U\,,$$

or, in matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ -\frac{1}{LC} \end{bmatrix}U = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,U\,.$$

II. Questions on system properties. The objective of the following set of problems is to determine what properties a system has. Here, you just have to check the definition of these properties.

1. (6 points) Show that the system with excitation $x(t)$ and response $y(t)$ described by

$$y(t) = \cos(2\pi t)x(t)$$

is linear but time-variant, and BIBO stable. Is the system memoryless?

Solution:
Homogeneity:
Let $x_1(t) = g(t)$. Then $y_1(t) = \cos(2\pi t)g(t)$.
Let $x_2(t) = Kg(t)$. Then $y_2(t) = \cos(2\pi t)(Kg(t)) = K(\cos(2\pi t)g(t)) = Ky_1(t)$. This is true for any $g(t)$ and any K. Thus the system is homogeneous.

Additivity:
Let $x_1(t) = g_1(t)$. Then $y_1(t) = \cos(2\pi t)g_1(t)$.
Let $x_2(t) = g_2(t)$. Then $y_1(t) = \cos(2\pi t)g_2(t)$.
Let $x_3(t) = x_1(t)+x_2(t) = g_1(t)+g_2(t)$. Then $y_3(t) = \cos(2\pi t)(g_1(t)+g_2(t)) = \cos(2\pi t)g_1(t)+\cos(2\pi t)g_2(t) = y_1(t) + y_2(t)$. Thus the system is additive.

Since the system is both homogeneous and additive, it is also linear.

Time invariance:
Let $x_1(t) = g(t)$. Then $y_1(t) = \cos(2\pi t)g(t)$.
Let $x_2(t) = g(t - t_0)$. Then $y_2(t) = \cos(2\pi t)g(t - t_0) \neq \cos(2\pi(t - t_0))g(t - t_0)$. for example, for $t_0 = \frac{1}{4}$ and $g(t) = 1$, we have that $\cos(2\pi t - \frac{\pi}{2}) \neq \cos(2\pi t)$. Thus the system is time variant.

Stability:
Let $x_1(t) = g(t)$ bounded. This means that there is a constant $A \geq 0$ such that $-A \leq g(t) \leq A$. Since $-1 \leq \cos(2\pi t) \leq 1$, we have that $-A \leq \cos(2\pi t)g(t) \leq A$. Therefore $y_1(t)$ is also bounded. By definition, the system is BIBO stable.

Memory:
The value of the output $y(t) = \cos(2\pi t)x(t)$ only depends on the present value of the input $x(t)$, thus the system is memoryless.

2. (5 points) Show that the system with excitation $x(t)$ and response $y(t)$ described by

$$y(t) = \frac{1}{x(t)}u(t)$$

is neither homogeneous nor linear, and is time variant. Show also that the system is not BIBO stable.

Solution:
Homogeneity:
Let $x_1(t) = g(t)$. Then $y_1(t) = \frac{1}{g(t)}u(t)$.

Let $x_2(t) = Kg(t)$. Then $y_2(t) = \frac{1}{Kg(t)}u(t) = \frac{1}{K}y_1(t) \neq Ky_1(t)$ for some K and signals $x_1(t)$. For example, if we take a $K = 2$, and $x_1(t) = 1$ then we have $y_2(t) = \frac{1}{2} \neq 2y_1(t) = 2$.

Additivity:
Let $x_1(t) = g_1(t)$. Then $y_1(t) = \frac{1}{g_1(t)}u(t)$.
Let $x_2(t) = g_2(t)$. Then $y_1(t) = \frac{1}{g_2(t)}u(t)$.
Let $x_3(t) = x_1(t) + x_2(t) = g_1(t) + g_2(t)$. Then $y_3(t) = \frac{1}{g_1(t)+g_2(t)}u(t) \neq \frac{1}{g_1(t)}u(t) + \frac{1}{g_2(t)}u(t)$.
For example, take $x_1(t) = x_2(t) = 1$ and The system is not additive.

Time invariance:
Let $x_1(t) = g(t)$. Then $y_1(t) = \frac{1}{g(t)}u(t)$.
Let $x_2(t) = g(t - t_0)$. Then $y_2(t) = \frac{1}{g(t-t_0)}u(t) \neq \frac{1}{g(t-t_0)}u(t - t_0)$. Thus the system is time variant.

Stability:
Let $x_1(t) = 0$, which is clearly bounded. However $y_1(t) = +\infty$, thus the system can not be BIBO stable. Another input that works is $x(t) = \frac{1}{t}$. We have that $y(t) = \frac{1}{\frac{1}{t}}u(t) \rightarrow +\infty$ when $t \rightarrow 0$. Thus from the bounded input $x(t)$ we get an unbounded output $y(t)$.

Memory:
The value of the output $y(t)$ only depends on the present value of the input $x(t)$, thus the system is memoryless.

3. (2 points) Give an example of a memoryless system and another of a system with memory.

Solution: A system that has memory is the following. For every excitation $x(t)$, the output is $y(t) = x(t - 1)$. Here, the output value at the present time t depends on the value of the input at a past time, $x(t - 1)$. For example, $y(1)$ depends on $x(0)$.

A system that has no memory is the following. For every excitation $x(t)$, the output is $y(t) = (t - 1)^2 x(t)$. Here, the output value at the present time, $y(t)$, depends on the value of the input at the present time, $x(t)$. For example, $y(1)$ depends on $x(1)$ and so on. The coefficient $(t - 1)^2$ refers to a past time, but it does not affect the property of memory.

III. Questions on system responses, stability and memory (time domain approach). The objective of these problems is to obtain some system responses by either (i) using the LTI property of the system, (ii) solving an ODE, or (iii) using convolution and the impulse response. You could be asked to use properties of convolution, and to determine some system properties such as stability and memory. In particular, you should know how to determine these properties from the shape of the impulse response OR the shape of the step response; see exercises 2 and 5 of Homework 3.

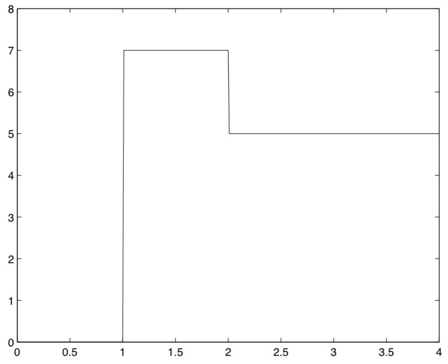

Figure 8: Circuit diagram for Problem III.1

1. (7 points) Let $U(t)$ be a periodic signal over $t \geq 0$ with fundamental period $T = 4$. For $t \in [0, 4]$ the signal is graphed in Figure 8.
   (i) Describe this signal for $t > 0$ as an infinite sum of shifted and amplified unit step functions.
   (ii) Suppose that the zero-state response of the circuit system to the unit-step function $u(t)$ is
   $$V(t) = \frac{1}{14}\mathrm{e}^{-\frac{t}{2}}u(t) + \frac{1}{7}u(t)\,,$$
   for certain values of R, L, C. Knowing that the circuit is an LTI system, determine the response $W(t)$ to this $U(t)$ from an initial condition at the zero state.

Solution: (i) The periodic signal of the figure, can be described as the following infinite sum:

$$\begin{aligned} U(t) = {} & 7\cdot u(t-1) - 2\cdot u(t-2) - 5\cdot u(t-4) \\ & + 7\cdot u(t-5) - 2\cdot u(t-6) - 5\cdot u(t-8) \\ & + 7\cdot u(t-9) - 2\cdot u(t-10) - 5\cdot u(t-12)\ldots \end{aligned}$$

or in a compact formula as:

$$U(t) = \sum_{k=0}^{+\infty} 7\cdot u(t-(4k+1)) - 2\cdot u(t-(4k+2)) - 5\cdot u(t-(4k+4))\,.$$

(ii) Because a RCL circuit can always be described by linear ODEs, the system is LTI. Therefore, the system response to the input $U(t)$ can be obtained as a sum of the following responses:

$$\begin{array}{rcl} \text{Input} & & \text{Output} \\ 7\cdot u(t-4k-1) & \rightarrow & 7\cdot V(t-4k-1)\,, \\ -2\cdot u(t-4k-2) & \rightarrow & -2\cdot V(t-4k-2)\,, \\ -5\cdot u(t-4k-3) & \rightarrow & -5\cdot V(t-4k-4)\,. \end{array}$$

Using the expression of $V(t) = \left(\frac{1}{14}\mathrm{e}^{-\frac{t}{2}} + \frac{1}{7}\right) u(t)$ we get:

$$\begin{aligned} 7 \cdot V(t-4k-1) &= \left(\frac{1}{2}\mathrm{e}^{-\frac{t-4k-1}{2}} + 1\right) u(t-4k-1)\,, \\ -2 \cdot V(t-4k-2) &= -\left(\frac{1}{7}\mathrm{e}^{-\frac{t-4k-2}{2}} + \frac{2}{7}\right) u(t-4k-2)\,, \\ -5 \cdot V(t-4k-4) &= -\left(\frac{5}{14}\mathrm{e}^{-\frac{t-4k-4}{2}} + \frac{5}{7}\right) u(t-4k-4)\,. \end{aligned}$$

In this way, the overall response of the system to $U(t)$ will be:

$$\begin{aligned} W(t) = \sum_{k=0}^{+\infty} &\left(\frac{1}{2}\mathrm{e}^{-\frac{t-4k-1}{2}} + 1\right) u(t-4k-1) - \left(\frac{1}{7}\mathrm{e}^{-\frac{t-4k-2}{2}} + \frac{2}{7}\right) u(t-4k-2) \\ &- \left(\frac{5}{14}\mathrm{e}^{-\frac{t-4k-4}{2}} + \frac{5}{7}\right) u(t-4k-4)\,. \end{aligned}$$

2. (8 points) Suppose a system is modeled by

$$\ddot{y}(t) - \sqrt{8}\dot{y}(t) + y(t) = 2x(t)$$

Determine the set of zero-input responses of the system. What is the zero-input response to the system with initial condition $y(0) = C$, $\dot{y}(0) = D$? Is the system BIBO stable?

Solution: The set of zero-input responses of the system is the solution to the homogeneous equation

$$\ddot{y}(t) - \sqrt{8}\dot{y}(t) + y(t) = 0$$

with general initial conditions. In this case, the solution of the ODE has the form $y(t) = K_1\,\mathrm{e}^{\lambda_1 t} + K_2\,\mathrm{e}^{\lambda_2 t}$, where λ_1, λ_2 are simple eigenvalues of the system. The eigenvalues can be obtained from the equation:

$$\lambda(\lambda - \sqrt{8}) + 1 = 0\,.$$

The solutions to this equation are:

$$\lambda = \frac{\sqrt{8} \pm \sqrt{8 - 4 \cdot 1}}{2} = \frac{2\sqrt{2} \pm 2}{2} = \sqrt{2} \pm 1\,.$$

In other words, $\lambda_1 = \sqrt{2} + 1$ and $\lambda_2 = \sqrt{2} - 1$. From here, we have that the set of zero-input responses to the system is $y(t) = K_1\mathrm{e}^{(\sqrt{2}+1)t} + K_2\mathrm{e}^{(\sqrt{2}-1)t}$.

Now suppose we have special initial conditions. The initial conditions determine the constants K_1, K_2 as follows:

$$\begin{aligned} C &= y(0) = K_1 + K_2 \\ D &= \dot{y}(0) = K_1(\sqrt{2}+1) + K_2(\sqrt{2}-1)\,. \end{aligned}$$

Solving for K_1 and K_2, we get:

$$K_1 = \frac{C(1-\sqrt{2})+D}{2},$$
$$K_2 = \frac{C(\sqrt{2}+1)-D}{2}.$$

That is, the solution is $y(t) = \frac{C(1-\sqrt{2})+D}{2}\mathrm{e}^{(\sqrt{2}+1)t} + \frac{C(\sqrt{2}+1)-D}{2}K_2\mathrm{e}^{(\sqrt{2}-1)t}$.

The set of zero-input responses do not satisfy the condition of BIBO stability because the exponentials in the general solution have positive real parts. Thus the system is not BIBO stable.

3. (4 points) Find the impulse response of the system described by the following equation.

$$5\dot{y}(t) + 4y(t) = x(t)$$

Is the system BIBO stable? If the answer is affirmative, what is the approximate memory of the system?

Solution: The impulse response $h(t)$ just satisfies:

$$5\dot{h}(t) + 4h(t) = \delta(t)$$

We can use a time-domain approach to solve for $h(t)$ or use a Laplace transform method. Let us follow the time-domain approach here.

We have that $h(t) = y_h(t)u(t)$, where $u(t)$ is the unit step function and $y_h(t)$ is the solution to the homogenous equation. A quick calculation can show that $\lambda = -\frac{4}{5}$ is an eigenvalue for the ODE and that $y_h(t) = \mathrm{e}^{-\frac{4}{5}t}$. Therefore $h(t) = K\mathrm{e}^{-\frac{4}{5}t}u(t)$. To determine the value of K, let us take the convention that $\dot{\mu} = \delta$ and substitute $y_h(t)\mu(t)$ into the ODE. We have

$$5\dot{y}_h\mu + 5y_h\dot{\mu} + 4y_h\mu = \delta \qquad \Longrightarrow \qquad 5y_h\dot{\mu} = 5y_h\delta = \delta$$

Taking in the last equation $t = 0$, we obtain

$$5K\mathrm{e}^0 = 1 \qquad \Longrightarrow \qquad K = \frac{1}{5}$$

Thus $h(t) = \frac{1}{5}\mathrm{e}^{-\frac{5}{4}t}u(t)$.

The system is BIBO stable because $\int_{-\infty}^{\infty} |\frac{1}{5}\mathrm{e}^{-\frac{5}{4}t}u(t)|dt < +\infty$.

The settling time of the system is approximately t_s that satisfies $\frac{1}{5}\mathrm{e}^{-\frac{5}{4}t_s} = 0.01$. In other words, $t_s = -4\frac{\log 0.05}{5}$. This tells us that the system memory is of the order of $-4\frac{\log 0.05}{5}$ seconds. (Here log means natural logarithm.)

4. (6 points) Find the values to these functions using the convolution operation and its properties.

(i) $g(t) = 4\sin(\frac{\pi t}{8}) * \delta(t-4)$ and $g(8)$.
(ii) $g(t) = \text{ramp}(t) * [\delta(t+2) - \delta(t+1)]$ and $g(4)$.
(iii) $g(t) = \mathrm{e}^{-t}u(t) * \mathrm{e}^{-t}u(t)$, $g(2)$, and $g(-1)$.

Solution:

(i) Using the sampling property of δ, we have that $g(t) = 4\sin(\frac{\pi t}{8}) * \delta(t-4) = 4\sin(\frac{\pi(t-4)}{8})$, therefore $g(8) = 4\sin(\frac{\pi(8-4)}{8}) = 4\sin(\frac{\pi}{2}) = 4$

(ii) Using the distribute property of $*$ we have that $g(t) = \text{ramp}(t)*\delta(t+2) - \text{ramp}(t)*\delta(t+1)$. Now using the sampling property of δ we have that $g(t) = \text{ramp}(t+2) - \text{ramp}(t+1)$. From here, we have that $g(4) = \text{ramp}(6) - \text{ramp}(5) = 6 - 5 = 1$.

(iii) To solve this part, we use the definition of convolution. We have that:

$$g(t) = \mathrm{e}^{-t}u(t) * \mathrm{e}^{-t}u(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\tau}u(\tau)\mathrm{e}^{-(t-\tau)}u(t-\tau)d\tau$$
$$\left(\int_0^t \mathrm{e}^{-\tau}\mathrm{e}^{-(t-\tau)}d\tau\right)u(t) = \left(\int_0^t \mathrm{e}^{-\tau-t+\tau}d\tau\right)u(t) = t\mathrm{e}^{-t}u(t)$$

Then, $g(2) = 2\mathrm{e}^{-2}$ and $g(-1) = 0$.

5. (6 points) An LTI system has an impulse response $h(t) = 4\mathrm{e}^{-4t}u(t)$. Suppose the system is initially at the zero state.

(i) What is the system response to the excitation $x(t) = u(t) - u(t-\frac{1}{2})$? Graph the output to $x(t)$.
(ii) Suppose the system is connected in series with another with impulse response $h_2(t) = \delta(t-\frac{1}{2})$. What is the step response to the new cascaded system?

Solution:

(i) The system response can be computed using the convolution formula. Since $u(t)$ is the unit step, its convolution with the impulse response is just the integral of the impulse response. That is,

$$y_1(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)d\tau = (4 - \mathrm{e}^{-4t})u(t)$$

The system is LTI, thus the response to $u(t-\frac{1}{2})$ will be $y_1(t-\frac{1}{2})$. In this way, the overall system response is:

$$y(t) = (4 - \mathrm{e}^{-4t})u(t) - (4 - \mathrm{e}^{-4t+2})u(t - -\frac{1}{2})$$

(ii) The impulse response to the cascaded system is $h_c(t) = h(t) * \delta(t - \frac{1}{2})$. Then the step response is $y_s(t) = u(t) * h_c(t) = \int_{-\infty}^{t} h_c(\tau) d\tau$. First, we have that:

$$h_c(t) = 4\mathrm{e}^{-4t} u(t) * \delta(t - \frac{1}{2}) = 4\mathrm{e}^{-4t+2} u(t - \frac{1}{2})$$

From here, we obtain:

$$\begin{aligned} y_s(t) &= \int_{-\infty}^{t} h_c(\tau) d\tau = \int_{-\infty}^{t} 4\mathrm{e}^{-4\tau+2} u(\tau - \frac{1}{2}) d\tau \\ &= \left(\int_{\frac{1}{2}}^{t} 4\mathrm{e}^{-4\tau+2} d\tau \right) u(t - \frac{1}{2}) = \mathrm{e}^{2}(\mathrm{e}^{-2} - \mathrm{e}^{-4t}) u(t - \frac{1}{2}) \\ &= (1 - \mathrm{e}^{-4t+2}) u(t - \frac{1}{2}) \end{aligned}$$