Signals and Systems - MAE 143A

Midterm Exam - Winter Quarter 2008

Student name and number ___________________________________

For all the questions you need to show ALL your work to get to the answer.

1. (10 points) The circuit in Figure 1 (left graph) represents a "practical passive filter". The circuit is excited by a voltage difference $U(t)$ and has a response of $V(t)$. Determine:

   (i) A model of the circuit in the form of a second-order ODE. *Hint:* Take derivatives with respect to time in the capacitor law and combine it with the inductor law.

   (ii) The state-space representation of the system. Write the state-space model in matrix form.

Solution:

(i) An ODE model for this system can be found using the the KVL and circuit element laws. Let i be the current going through the resistor, inductor and capacitor. Let V_R, V_L and V_C be the voltage differences across R, L, and C, respectively.

By the KCL, we have that $i_R = i_C = i_L$.

By the KVL, we have that:

$$U - V_R - V_L - V_C = 0\,, \qquad \text{using KVL for the marked loop}\,, \tag{1}$$

$$V_C - V = 0\,, \qquad \text{using also KVL for the loop with } C \text{ and } V \tag{2}$$

The circuit element laws are given by:

$$V_R = iR\,, \tag{3}$$

$$L\frac{di}{dt} = V_L \quad \Longleftrightarrow \quad \frac{di}{dt} = \frac{1}{L}V_L \tag{4}$$

$$i = C\frac{dV_C}{dt}\,, \tag{5}$$

By equation (2), we have that $V_C = V$. Since everything has to be expressed in terms of V, we differentiate with respect to time in equation (5). Putting this together with equation (4), we obtain V_L in terms of the second derivative of V:

$$\begin{cases} \dfrac{di}{dt} = C\dfrac{d^2V}{dt^2} \\ \\ \dfrac{di}{dt} = \dfrac{1}{L}V_L \end{cases} \quad \Longrightarrow \quad V_L = LC\,\frac{d^2V}{dt^2}$$

On the other hand, (3) and (5) give that $V_R = CR\,\dfrac{dV}{dt}$. Now, using equation (1), and substituting the obtained values of V_L and V_R, we have that:

$$U = CR\frac{dV}{dt} + LC\frac{d^2V}{dt^2} + V \tag{6}$$

(ii) There is only one unknown in the system and that is the response V. Since the ODE is second order, the state dimension will be 2. Let us denote by $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ the state vector variable. We have that:

$$x_1 = V\,, \qquad x_2 = \dot{V}\,.$$

Then, $\dot{x}_1 = x_2$ and, using equation (6), $\dot{x}_2 = -\frac{1}{LC}V - \frac{CR}{LC}\dot{V} - \frac{1}{LC}U$. From here we have:

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= -\frac{1}{LC}x_1 - \frac{R}{L}x_2 - \frac{1}{LC}U\,, \end{aligned}$$

or, in matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ -\frac{1}{LC} \end{bmatrix}U = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,U\,.$$

2. (7 points) Let $U(t)$ be a periodic signal defined for all $t \in \mathbb{R}$ with fundamental period $T = 4$. For $t \in [0, 4]$ the signal is graphed in Figure 1 (right).

   (i) Describe this signal for $t > 0$ as an infinite sum of shifted and amplified unit step functions.

   (ii) Suppose that the zero-state response of the circuit system to the unit step function $1(t)$ is

   $$V(t) = \frac{1}{14}\mathrm{e}^{-\frac{t}{2}} + \frac{1}{7}\,, \quad \text{for } t > 0\,,$$

   for certain values of R, L, C. Determine the circuit system response $W(t)$ to this $U(t)$ from an initial condition at the zero state. Justify your answer based on the system properties.

Solution: (i) The periodic signal of the figure, can be described as the following infinite sum:

$$\begin{aligned} U(t) = {} & 7\cdot 1(t-1) - 2\cdot 1(t-2) - 5\cdot 1(t-4) \\ & + 7\cdot 1(t-5) - 2\cdot 1(t-6) - 5\cdot 1(t-8) \\ & + 7\cdot 1(t-9) - 2\cdot 1(t-10) - 5\cdot 1(t-12)\ldots \end{aligned}$$

or in a compact formula as:

$$U(t) = \sum_{k=0}^{+\infty} 7\cdot 1(t-(4k+1)) - 2\cdot 1(t-(4k+2)) - 5\cdot 1(t-(4k+4))\,.$$

(ii) Because the state space representation of the circuit can be written in the form $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,U$ for constant matrices $\mathbf{A}$ and $\mathbf{B}$, the system is LTI. Therefore, the system response to the input $U(t)$ can be obtained as a sum of the following responses:

$$\begin{array}{rcl} \text{Input} & & \text{Output} \\ 7\cdot 1(t-4k-1) & \rightarrow & 7\cdot V(t-4k-1)\,, \\ -2\cdot 1(t-4k-2) & \rightarrow & -2\cdot V(t-4k-2)\,, \\ -5\cdot 1(t-4k-3) & \rightarrow & -5\cdot V(t-4k-4)\,. \end{array}$$

Using the expression of $V(t) = \left(\frac{1}{14}\mathrm{e}^{-\frac{t}{2}} + \frac{1}{7}\right) 1(t)$ we get:

$$
\begin{aligned}
7 \cdot V(t-4k-1) &= \left(\frac{1}{2}\mathrm{e}^{-\frac{t-4k-1}{2}} + 1\right) 1(t-4k-1)\,,\\
-2 \cdot V(t-4k-2) &= -\left(\frac{1}{7}\mathrm{e}^{-\frac{t-4k-2}{2}} + \frac{2}{7}\right) 1(t-4k-2)\,,\\
-5 \cdot V(t-4k-4) &= -\left(\frac{5}{14}\mathrm{e}^{-\frac{t-4k-4}{2}} + \frac{5}{7}\right) 1(t-4k-4)\,.
\end{aligned}
$$

In this way, the overall response of the system to $U(t)$ will be:

$$
\begin{aligned}
W(t) = \sum_{k=0}^{+\infty} &\left(\frac{1}{2}\mathrm{e}^{-\frac{t-4k-1}{2}} + 1\right) 1(t-4k-1) - \left(\frac{1}{7}\mathrm{e}^{-\frac{t-4k-2}{2}} + \frac{2}{7}\right) 1(t-4k-2)\\
&- \left(\frac{5}{14}\mathrm{e}^{-\frac{t-4k-4}{2}} + \frac{5}{7}\right) 1(t-4k-4)\,.
\end{aligned}
$$

3. (8 points) Suppose a system is modeled by

$$
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & \sqrt{8}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 2\end{bmatrix} u\,.
$$

Determine the zero-input response of the system to any initial condition $x_1(0) = C$, $x_2(0) = D$. Does the zero-input response satisfy the conditions for BIBO stability?

Solution: The zero-input response of the system is the solution to the homogeneous equation

$$
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & \sqrt{8}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}
$$

with initial conditions $x_1(0) = C$ and $x_2(0) = D$. The solution of the ODE has the form $x_1(t) = K_1\,\mathrm{e}^{\lambda_1 t} + K_2\,\mathrm{e}^{\lambda_2 t}$, where λ_1, λ_2 are the solutions to $\det(\lambda I_2 - \mathbf{A}) = 0$. That is,

$$
\det\begin{bmatrix}\lambda & -1\\ 1 & \lambda - \sqrt{8}\end{bmatrix} = 0\,, \qquad \Longleftrightarrow \qquad \lambda(\lambda - \sqrt{8}) + 1 = 0\,.
$$

The solutions to this equation are:

$$
\lambda = \frac{\sqrt{8} \pm \sqrt{8 - 4\cdot 1}}{2} = \frac{2\sqrt{2} \pm 2}{2} = \sqrt{2} \pm 1\,.
$$

In other words, $\lambda_1 = \sqrt{2} + 1$ and $\lambda_2 = \sqrt{2} - 1$. The solution to the homogeneous equation has the form $\mathbf{x} = K_1\mathbf{v}_1\mathrm{e}^{\lambda_1 t} + K_2\mathbf{v}_2\mathrm{e}^{\lambda_2 t}$. To determine $\mathbf{v}_1$ and $\mathbf{v}_2$, we assume without lose of generality $\mathbf{v}_1 = \begin{bmatrix}1\\ v_1\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}1\\ v_2\end{bmatrix}$. These eigenvectors must satisfy:

$$
\begin{aligned}
\begin{bmatrix}\lambda_1 & -1\\ 1 & \lambda_1 - \sqrt{8}\end{bmatrix}\begin{bmatrix}1\\ v_1\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\,,\\
\begin{bmatrix}\lambda_2 & -1\\ 1 & \lambda_2 - \sqrt{8}\end{bmatrix}\begin{bmatrix}1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\,.
\end{aligned}
$$

From here, it is easy to see that $v_1 = \lambda_1$ and $v_2 = \lambda_2$ and then:

$$x_1(t) = K_1 e^{(\sqrt{2}+1)t} + K_2 e^{(\sqrt{2}-1)t}\,,$$
$$x_2(t) = K_1(\sqrt{2}+1)e^{(\sqrt{2}+1)t} + K_2(\sqrt{2}-1)e^{(\sqrt{2}-1)t}\,.$$

The initial conditions determine the constants K_1, K_2 as follows:

$$C = x_1(0) = K_1 + K_2$$
$$D = x_2(0) = K_1(\sqrt{2}+1) + K_2(\sqrt{2}-1)\,.$$

Solving for K_1 and K_2, we get:

$$K_1 = \frac{C(1-\sqrt{2}) + D}{2}\,,$$
$$K_2 = \frac{C(\sqrt{2}+1) - D}{2}\,.$$

The solutions do not satisfy the condition of BIBO stability because the exponentials in $x_1(t)$ and $x_2(t)$ grow unbounded as t tends to infinity.

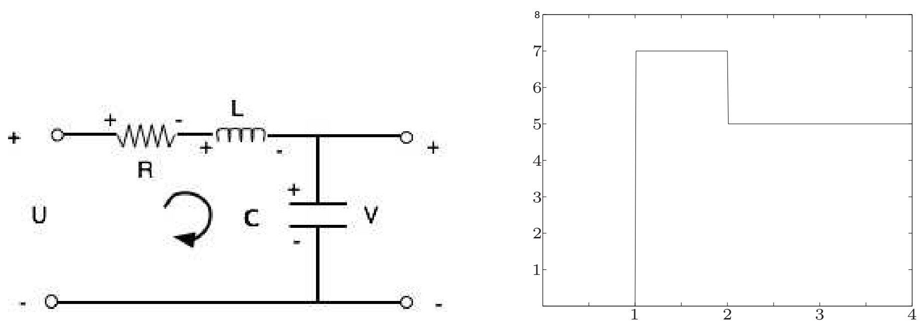

Figure 1: Circuit diagram for Problem 1 (left) and input signal for Problem 2 (right)