Signals and Systems - MAE 143A

Midterm Exam - Winter Quarter 2007

Student name and number ________________________________

For all the questions you need to show ALL your work to get to the answer.

1. (4.5 points) Consider the set of differential equations:

$$\dot{x} = x + 2z + u$$
$$\dot{z} = -z + 3x$$
$$y = 3z + 2x$$

where u is a known external signal and y is the output signal. (i) Find the state space representation of the system, and (ii) compute the transfer function of the system using the matrix formula. Recall that the inverse of a 2×2 matrix M is given as follows:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Solution: (i) The system is already described as a set of first order differential equations. The state components will be those unknown variables whose derivative appears in the system. And the input is given by the known external signal. In other words,

$$\mathbf{x} = \begin{bmatrix} x \\ z \end{bmatrix}, \qquad \mathbf{u} = u\,.$$

In terms of these, we have that

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} x + 2z + u \\ -z + 3x \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u\,.$$

That is, $\dot{\mathbf{x}} = F\mathbf{x} + G\mathbf{u}$, with

$$F = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The output equation $y = H\mathbf{x}$ is obtained from the third equation given in the statement of the problem, $y = [2 \;\; 3]\,\mathbf{x}$.

(ii) The transfer function matrix formula is $P(s) = H[sI_2 - F]^{-1}G$, where I_2 is the 2×2 identity matrix. We compute first the inverse of $[sI_2 - F]^{-1}$ according to the previous formula:

$$sI_2 - F = \begin{bmatrix} s-1 & -2 \\ -3 & s+1 \end{bmatrix}, \qquad [sI_2 - F]^{-1} = \frac{1}{(s-1)(s+1) - 6} \begin{bmatrix} s+1 & 2 \\ 3 & s-1 \end{bmatrix}.$$

From here,

$$P(s) = \frac{1}{s^2-7}[2\ \ 3]\begin{bmatrix} s+1 & 2 \\ 3 & s-1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{s^2-7}\left[2(s+1)+9, \ \ 4+3(s-1)\right]$$

$$= \left[\frac{2s+11}{s^2-7}, \ \frac{3s+1}{s^2-7}\right]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2s+11}{s^2-7}.$$

2. (4.5 points) Consider a system of nonlinear second order differential equations of the form:

$$l\ddot{\theta} + \sin\theta = u$$
$$m\dot{z} + z = uz\,.$$

where u is the input, θ, z are unknown variables that describe the behavior of the system, θ is the output and l and m are system constants.

(i) Obtain a state-space representation of the system

(ii) Determine the conditions that equilibrium points have to satisfy. Is $\theta = 0$, $\dot{\theta} = 0$, $z = 0$ and $u = 0$ an equilibrium for the system?

(iii) Linearize the equation about $\theta = 0$, $\dot{\theta} = 0$, $z = 0$ and $u = 0$.

Solution: (i) The system is composed of first and second order differential equations. In order to write this as a first order system, we define

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ z \end{bmatrix} \quad \Longrightarrow \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{z} \end{bmatrix}.$$

Rewriting the equations in terms of the state we have that:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{l}\left(-\sin x_1 + u\right)$$
$$\dot{x}_3 = \frac{1}{m}\left(ux_3 - x_3\right)$$

or, in other words,

$$\dot{\mathbf{x}} = f(\mathbf{x}, u) = \begin{bmatrix} f_1(\mathbf{x}, u) \\ f_2(\mathbf{x}, u) \\ f_3(\mathbf{x}, u) \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{l}\left(-\sin x_1 + u\right) \\ \frac{1}{m}\left(ux_3 - x_3\right) \end{bmatrix}.$$

(ii) A point $(\mathbf{x}_0, u_0)$ is an equilibrium for the system if $f(\mathbf{x}_0, u_0) = 0$. In other words,

$$\begin{aligned} &x_2 = 0 \\ &\frac{1}{l}(-\sin x_1 + u) = 0 \\ &\frac{1}{m}(ux_3 - x_3) = 0 \end{aligned}$$

The solution of these equations leads to $x_2 = 0$, $(u-1)x_3 = 0$ and $u = \sin x_1$. In particular, these equations are verified by $u = 0$, $z = 0$, $\dot{\theta} = 0$ and $\theta = 0$.

(iii) The linearization of the system about the equilibrium point is the following.

$$\dot{\mathbf{x}} = F(\mathbf{x} - \mathbf{x}_0) + G(u - u_0)\,,$$

where

$$F = \begin{bmatrix} \frac{\partial f_1(\mathbf{x},u)}{\partial x_1}_{|(\mathbf{x}_0,u_0)} & \frac{\partial f_1(\mathbf{x},u)}{\partial x_2}_{|(\mathbf{x}_0,u_0)} & \frac{\partial f_1(\mathbf{x},u)}{\partial x_3}_{|(\mathbf{x}_0,u_0)} \\ \frac{\partial f_2(\mathbf{x},u)}{\partial x_1}_{|(\mathbf{x}_0,u_0)} & \frac{\partial f_2(\mathbf{x},u)}{\partial x_2}_{|(\mathbf{x}_0,u_0)} & \frac{\partial f_2(\mathbf{x},u)}{\partial x_3}_{|(\mathbf{x}_0,u_0)} \\ \frac{\partial f_3(\mathbf{x},u)}{\partial x_1}_{|(\mathbf{x}_0,u_0)} & \frac{\partial f_3(\mathbf{x},u)}{\partial x_2}_{|(\mathbf{x}_0,u_0)} & \frac{\partial f_3(\mathbf{x},u)}{\partial x_3}_{|(\mathbf{x}_0,u_0)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{l}(\cos x_1)_{|(\mathbf{x}_0,u_0)} & 0 & 0 \\ 0 & 0 & \frac{1}{m}(u-1)_{|(\mathbf{x}_0,u_0)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1}{l} & 0 & 0 \\ 0 & 0 & \frac{-1}{m} \end{bmatrix}$$

and

$$G = \begin{bmatrix} \frac{\partial f_1(\mathbf{x},u)}{\partial u}_{|(\mathbf{x}_0,u_0)} \\ \frac{\partial f_2(\mathbf{x},u)}{\partial u}_{|(\mathbf{x}_0,u_0)} \\ \frac{\partial f_3(\mathbf{x},u)}{\partial u}_{|(\mathbf{x}_0,u_0)} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{l} \\ 0 \end{bmatrix}.$$

In other words, the linearized system becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1}{l} & 0 & 0 \\ 0 & 0 & \frac{-1}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{l} \\ 0 \end{bmatrix} u\,.$$

3. (6.5 points) Find the state space representation of the circuit shown in Figure 1 with input voltage U and output voltage V. Is the system linear? Is the system time-invariant? Is the system finite-dimensional? Justify your answers. *Hint:* The state is $\mathbf{x} = [i_L, V]^T$.

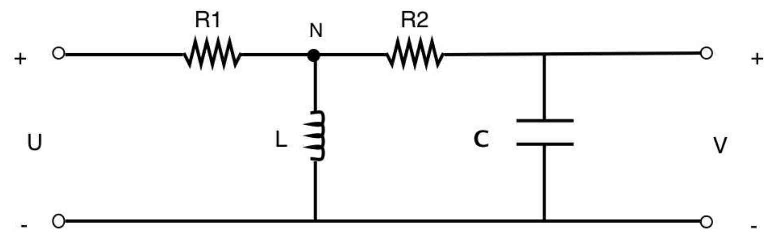

Figure 1: Circuit diagram for problem 3

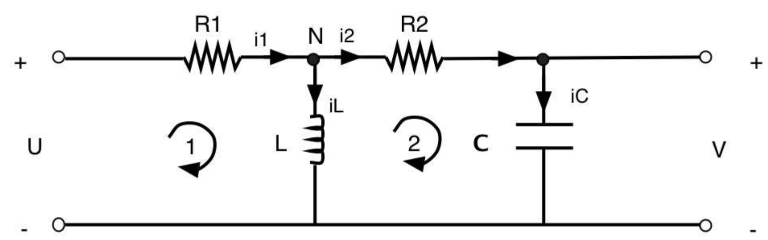

Figure 2: Worked out diagram for problem 3

Solution:

Consider the worked diagram in Figure 2. Let i_1 be the current passing through R_1, i_2 the current passing through R_2, i_L the current passing through L and i_C the current passing through C, respectively.

Applying KCL at N, we have:

$$i_1 = i_2 + i_L \tag{1}$$

By the KVL at loop 1:

$$U - i_1 R_1 - L\frac{di_L}{dt} = 0\,, \tag{2}$$

and the KVL at loop 2:

$$-\, i_2 R_2 - V + L\frac{di_L}{dt} = 0\,. \tag{3}$$

Using the element equation at the capacitor we get:

$$C\frac{dV}{dt} = i_2 = i_C\,. \tag{4}$$

We have four independent equations and four unknows V, i_1, i_2, i_L and therefore a closed system we can solve. We can simplify now the equations as follows. Using (1) and (4) we have

$$i_1 = C\frac{dV}{dt} + i_L\,. \tag{5}$$

Combining equation (5) and (2) leads to:

$$U - R_1C\frac{dV}{dt} - i_LR_1 - L\frac{di_L}{dt} = 0\,, \tag{6}$$

and the combination of equation (3) and (4),

$$-CR_2\frac{dV}{dt} - V + L\frac{di_L}{dt} = 0\,. \tag{7}$$

Equations (6) and (7) are a closed system with two unknowns i_L and V. In other words, we can describe the circuit by the system of equations:

$$\begin{aligned} L\frac{di_L}{dt} + R_1C\frac{dV}{dt} &= -i_LR_1 + U\,, \\ L\frac{di_L}{dt} - CR_2\frac{dV}{dt} - V &= 0\,. \end{aligned}$$

This is a state space representation of the system, with state given by $x = [i_L, V]^T$, input U and output V. In matrix form this can be written as:

$$\begin{bmatrix} L & CR_1 \\ L & -CR_2 \end{bmatrix} \begin{bmatrix} \frac{di_L}{dt} \\ \frac{dV}{dt} \end{bmatrix} = \begin{bmatrix} U - i_LR_1 \\ V \end{bmatrix} = \begin{bmatrix} -R_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_L \\ V \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U$$

The system is linear because the unknowns of the state space equations and their derivatives appear linearly in the equations. The system is time-invariant because the coefficients in the ODE system are constant. The system is finite-dimensional because it is only necessary to give two initial conditions, $i_L(0)$ and $V(0)$ to find the solutions.