

**Problem 1.** (15 points)

Consider the following system

$$\begin{aligned}\dot{\zeta} - 2\zeta &= \theta + 2\zeta u, \\ \ddot{\theta} + \theta^2(1 + \dot{\theta}) &= u + \theta.\end{aligned}$$

(This system does not arise from any physical application but its structure and its nonlinear terms mimic phenomena that appear in mechanical, electrical and bio-chemical systems).

- (a) (5 points) Treating  $\zeta$  as the output and  $u$  as the input, derive a state space representation of the system.
- (b) (5 points) Let  $u = 0$  and find *all* the equilibrium points of the state space system.
- (c) (5 points) For *each* equilibrium found in (b), compute the linearization of the system (in state space form, i.e., find  $F$ ,  $G$ ,  $H$ ,  $J$ ).

**Solution:**

- (a) Calling  $x_1 = \theta$ ,  $x_2 = \dot{\theta}$ , and  $x_3 = \zeta$ , we get

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1 - x_1^2(1 + x_2) + u, \\ \dot{x}_3 &= x_1 + 2x_3 + 2x_3u,\end{aligned}$$

and since  $\zeta$  is the output, then  $y = x_3$ . This can be written as

$$\begin{aligned}\dot{x} &= f(x, u), \\ y &= h(x, u),\end{aligned}$$

where

$$f(x, u) = \begin{bmatrix} x_2 \\ x_1 - x_1^2(1 + x_2) + u \\ x_1 + 2x_3 + 2x_3u \end{bmatrix}, \quad h(x, u) = x_3.$$

- (b) For finding the equilibria we set  $\dot{x} = 0$  and  $u = 0$ . Then, we need to solve the equation  $f(x, 0) = 0$ , i.e.,

$$\begin{aligned}0 &= x_2, \\ 0 &= x_1 - x_1^2(1 + x_2), \\ 0 &= x_1 + 2x_3.\end{aligned}$$

Hence,  $x_2 = 0$  for all equilibrium points, and from the second equation,  $x_1 - x_1^2 = 0$  which has two solutions,  $x_1 = 0$  and  $x_1 = 1$ . From the third equation,  $x_3 = -x_1/2$ . Therefore we find two equilibria:

$$E_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix}.$$

(c) We first compute the jacobian and the derivative of  $f(x, u)$  with respect to  $u$ :

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 - 2x_1(1 + x_2) & -x_1^2 & 0 \\ 1 & 0 & 2 + 2u \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 1 \\ 2x_3 \end{bmatrix}.$$

Since  $h$  is already linear, no further derivative needs to be computed.

For small displacements  $\delta x$  near the equilibrium  $E_1$  and  $\delta u$  near zero, the system can be approximated as the linear system

$$\begin{aligned} \delta \dot{x} &= F_1 \delta x + G_1 \delta u, \\ y &= H \delta x + J \delta u, \end{aligned}$$

where

$$F_1 = \left. \frac{\partial f}{\partial x} \right|_{x=E_1, u=0} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad G_1 = \left. \frac{\partial f}{\partial u} \right|_{x=E_1, u=0} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad J = [0].$$

For small displacements  $\delta x$  near the equilibrium  $E_2$  and  $\delta u$  near zero, the system can be approximated as the linear system

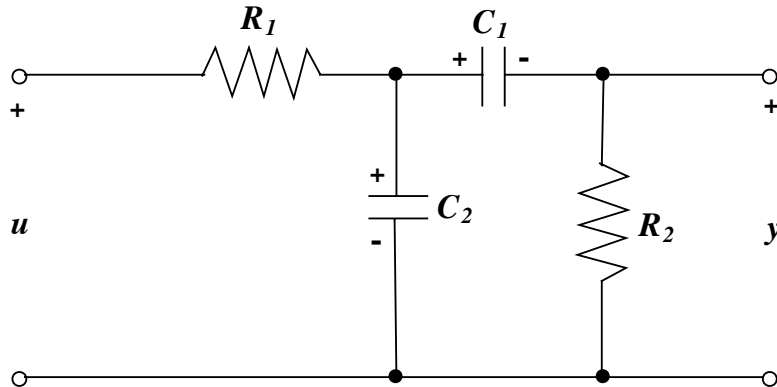
$$\begin{aligned} \delta \dot{x} &= F_2 \delta x + G_2 \delta u, \\ y &= H \delta x + J \delta u, \end{aligned}$$

where  $H$  and  $J$  are as above, and

$$F_2 = \left. \frac{\partial f}{\partial x} \right|_{x=E_2, u=0} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad G_2 = \left. \frac{\partial f}{\partial u} \right|_{x=E_2, u=0} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

**Problem 2.** (10 points)

Consider the following circuit



Treating  $u$  as the input and  $y$  as the output, derive a state space representation for the system.

**Solution:**

Call  $x_1$  and  $x_2$  the voltages across the capacitors  $C_1$  and  $C_2$ , respectively, according to the polarity in the figure. Applying KCL in the two nodes of the circuit, we get the following equations:

$$\begin{aligned} \frac{u - x_2}{R_1} &= C_1 \dot{x}_1 + C_2 \dot{x}_2, \\ C_1 \dot{x}_1 &= \frac{x_2 - x_1}{R_2}. \end{aligned}$$

From KVL we get  $y = x_2 - x_1$ . Manipulating the above equations we get an state space representation of the system as

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{R_2 C_1} x_1 + \frac{1}{R_2 C_1} x_2, \\ \dot{x}_2 &= \frac{1}{R_2 C_2} x_1 - \left( \frac{1}{R_2 C_2} + \frac{1}{R_1 C_2} \right) x_2 + \frac{1}{R_1 C_2} u, \\ y &= -x_1 + x_2. \end{aligned}$$

Since the system is linear it can be written as

$$\begin{aligned} \dot{x} &= Fx + Gu, \\ y &= Hx + Ju, \end{aligned}$$

where

$$F = \begin{bmatrix} -\frac{1}{R_2 C_1} & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\left( \frac{1}{R_2 C_2} + \frac{1}{R_1 C_2} \right) \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \frac{1}{R_1 C_2} \end{bmatrix}, \quad H = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad J = [0].$$