

Signals and Systems - MAE 143A

Practice Final Exam- Winter Quarter 2007

Student name and number _____

For all the questions you need to show ALL your work to get to the answer.

1. (4 points) For the time function

$$f(t) = \int_0^t \tau \sin(2(t - \tau)) d\tau, \quad t \geq 0$$

- (i) (2 points) Find the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$.
(ii) (2 points) Using the Initial Value Theorem, find $f(0)$.

Solution:

- (i) Observe that $f(t)$ is just a convolution $f(t) = t * \sin 2t$. Therefore its Laplace transform can be computed as follows:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} \cdot \mathcal{L}\{\sin 2t\} = \frac{1}{s^2} \cdot \frac{2}{s + 4}.$$

- (ii) The Initial Value Theorem can be applied for any function $F(s)$, and it says that:

$$f(0) = \lim_{s \rightarrow \infty} sF(s).$$

In our case, this leads to:

$$f(0) = \lim_{s \rightarrow \infty} \frac{2s}{s^2(s + 4)} = 0.$$

Note: For the exam you have to know the Laplace transform properties, recognize a convolution, and know how to use the Initial Value Theorem and the Final Value Theorem.

2. (3 points) Given that

$$e^{-4t}1(t) \xleftrightarrow{\mathcal{L}} F(s)$$

find the inverse Laplace transforms of:

- (i) $F\left(\frac{s}{3}\right)$,
(ii) $F(s - 2) + F(s + 3)$,
(iii) $\frac{F(s)}{s}$.

Solution: In each of these, we have to use the properties of the Laplace transform.

(i) By the time scaling property,

$$f(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} F\left(\frac{s}{a}\right).$$

Then,

$$F\left(\frac{s}{3}\right) \xleftrightarrow{\mathcal{L}^{-1}} 3f(3t) = 3e^{-12t}1(3t) = 3e^{-12t}1(t).$$

(ii) By the frequency shift property,

$$e^{-at}f(t) \xleftrightarrow{\mathcal{L}} F(s+a).$$

Then,

$$F(s-2) + F(s+3) \xleftrightarrow{\mathcal{L}^{-1}} e^{2t}f(t) + e^{-3t}f(t) = e^{-2t}1(t) + e^{-7t}1(t).$$

(iii) By the integration property,

$$\int_0^t f(\tau) d\tau \xleftrightarrow{\mathcal{L}} \frac{F(s)}{s}.$$

Then,

$$\frac{F(s)}{s} \xleftrightarrow{\mathcal{L}^{-1}} \int_0^t f(\tau) d\tau = \int_0^t e^{-4\tau} d\tau = \frac{1}{4}(1 - e^{-4t}).$$

Note: For the exam you need to know how to use the Laplace transform properties (superposition, time shift, time scaling, frequency shift, convolution ...)

3. (4 points) Suppose that a system has a transfer function of the form:

$$P(s) = \frac{s-1}{(s+2)(s+a)},$$

for some $a \neq -2$. Knowing that we can expand $P(s)$ in fractions as:

$$P(s) = \frac{A}{s+2} + \frac{3}{s+a},$$

(i) find the numerical values of A and a ,

(ii) compute the *impulse* response to the system. Is the impulse response signal *stable*?

Solution:

- (i) To find the numerical values of A and a we use the partial fraction expansion method corresponding to different poles $p_1 = -2$ and $p_2 = -a$. We have that:

$$(s + 2)P(s)\big|_{s=-2} = A \iff \frac{s - 1}{s + a}\big|_{s=-2} = \frac{-3}{-2 + a} = A \quad (1)$$

and

$$(s + a)P(s)\big|_{s=-a} = 3 \iff \frac{s - 1}{s + 2}\big|_{s=-a} = \frac{-a - 1}{-a + 2} = 3 \quad (2)$$

Equations (1) and (2) form a system of independent equations and we can use them to find that $a = \frac{7}{2}$ and $A = -2$.

- (ii) The impulse response to the system is by definition $p(t) = \mathcal{L}^{-1}\{P(s)\}$. Using the above partial fraction expansion with the obtained numerical values, we have that:

$$p(t) = \mathcal{L}^{-1}\left\{\frac{-2}{s + 2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s + \frac{7}{2}}\right\} = -2e^{-2t}1(t) + 3e^{-\frac{7}{2}t}1(t).$$

As we can see, $\lim_{t \rightarrow \infty} p(t) = 0$. Since the limit is a finite value, the signal is stable by definition.

Note: You have to know the partial fraction expansion method very well and how to compute standard system responses. You have to know when a signal is stable or unstable.

4. (5 points) Find the *step* response to the system:

$$P(s) = \frac{s + 1}{s^2 + 6s + 9}.$$

Solution: The step response to $P(s)$ is obtained as $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, where

$$Y(s) = P(s)U(s) \quad U(s) = \mathcal{L}\{1(t)\} = \frac{1}{s}.$$

That is,

$$Y(s) = P(s)\frac{1}{s} = \frac{s + 1}{s(s^2 + 6s + 9)}.$$

To obtain the inverse Laplace transform of $Y(s)$, we use the partial fraction expansion method. First observe that $s^2 + 6s + 9 = (s + 3)^2$. The poles of $Y(s)$ are then $p_1 = -3$ (repeated) and $p_2 = 0$. Since $\deg(s + 1) < \deg(s(s + 3)^2)$, we can write:

$$Y(s) = \frac{c_1}{s + 3} + \frac{c_2}{(s + 3)^2} + \frac{c_3}{s}.$$

To find the constants c_i , we use Case 2 of the method. In this way,

$$c_2 = (s+3)^2 Y(s)|_{s=-3} = \frac{s+1}{s} \Big|_{s=-3} = \frac{2}{3},$$

$$c_1 = \frac{d}{ds} ((s+3)^2 Y(s)) \Big|_{s=-3} = \frac{d}{ds} \left(\frac{s+1}{s} \right) \Big|_{s=-3} = \frac{-1}{s^2} \Big|_{s=-3} = \frac{-1}{9},$$

$$c_3 = sY(s)|_{s=0} = \frac{s+1}{(s+3)^2} \Big|_{s=0} = \frac{1}{9}.$$

From here,

$$Y(s) = \frac{-1}{9(s+3)} + \frac{2}{3(s+3)^2} + \frac{1}{9s},$$

and the step response to the system becomes:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{9}e^{-3t}1(t) + \frac{2}{3}te^{-3t}1(t) + \frac{1}{9}1(t).$$

5. (2 points) For the system with transfer function:

$$P(s) = \frac{s+1}{s^2(s+3)}$$

find the differential equation governing the relationship between inputs $u(t)$ and outputs $y(t)$. (Assume zero initial conditions for $y(t)$ and $u(t)$.)

Solution: We have that:

$$Y(s) = \frac{s+1}{s^2(s+3)}U(s),$$

for any $Y(s) = \mathcal{L}\{y(t)\}$, and $U(s) = \mathcal{L}\{u(t)\}$. This implies:

$$(s^3 + 3s^2)Y(s) = (s+1)U(s) \quad \implies \quad \ddot{y} + 3\dot{y} = \dot{u} + u.$$

Note: You need to know how to pass from a Time Domain system representation to a Frequency Domain representation and vice versa. This applies for both continuous-time and discrete-time systems.

6. (9 points) A system is described by the ODE:

$$\ddot{x}(t) - \dot{x}(t) = u(t).$$

Using the Laplace transform method, find the solution to the system with initial conditions $x(0) = 1$, $\dot{x}(0) = 0$ and driven by the input $u(t) = \sin \pi t$.

Solution: The Laplace transform of the above ODE with the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$ and the input $u(t)$ is given as:

$$\begin{aligned} s^2 X(s) - s - sX(s) + 1 &= \mathcal{L}\{\sin \pi t\} = \frac{\pi}{s^2 + \pi^2} && \iff \\ s(s-1)X(s) - (s-1) &= \frac{\pi}{s^2 + \pi^2}. \end{aligned}$$

In other words,

$$X(s) = \frac{\pi}{s(s-1)(s^2 + \pi^2)} + \frac{1}{s}.$$

The solution is obtained as $x(t) = \mathcal{L}^{-1}\{X(s)\}$. To find its expression, we make use of the partial fraction expansion method. That is, we expand

$$Y(s) = \frac{\pi}{s(s-1)(s^2 + \pi^2)} = \frac{c_1}{s} + \frac{c_2}{s-1} + \frac{c_3}{s-j\pi} + \frac{c_4}{s+j\pi},$$

which is the first fraction in $X(s)$, and substitute the result in $X(s)$. The roots are all different, so the constants can be computed as follows:

$$\begin{aligned} c_1 &= sY(s)|_{s=0} = \frac{\pi}{(s-1)(s^2 + \pi^2)} \Big|_{s=0} = \frac{\pi}{(-1)(\pi^2)} = -\frac{1}{\pi}, \\ c_2 &= (s-1)Y(s)|_{s=1} = \frac{\pi}{s(s^2 + \pi^2)} \Big|_{s=1} = \frac{\pi}{1 + \pi^2}, \\ c_3 &= (s-j\pi)Y(s)|_{s=j\pi} = \frac{\pi}{s(s-1)(s+j\pi)} \Big|_{s=j\pi} \\ &= \frac{1}{2\pi(1-j\pi)} = \frac{1+j\pi}{2\pi(1-j\pi)(1+j\pi)} = \frac{1}{2\pi(1+\pi^2)} + j\frac{1}{2(1+\pi^2)}, \\ c_4 &= \bar{c}_3 = \frac{1}{2\pi(1+\pi^2)} - j\frac{1}{2(1+\pi^2)}. \end{aligned}$$

From here, we find a partial fraction expansion of $X(s)$ given as:

$$\begin{aligned} X(s) &= \left(\frac{\pi-1}{\pi}\right) \frac{1}{s} + \left(\frac{\pi}{1+\pi^2}\right) \frac{1}{s-1} + \\ &+ \left(\frac{1}{2\pi(1+\pi^2)} + j\frac{1}{2(1+\pi^2)}\right) \frac{1}{s-j\pi} + \left(\frac{1}{2\pi(1+\pi^2)} - j\frac{1}{2(1+\pi^2)}\right) \frac{1}{s+j\pi}. \end{aligned}$$

The inverse Laplace transform of $X(s)$ and the solution to the ODE is then:

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{X(s)\} = \\ &\left(\frac{\pi-1}{\pi}\right) 1(t) + \left(\frac{\pi}{1+\pi^2}\right) e^t 1(t) \\ &+ \left(\frac{1}{2\pi(1+\pi^2)} + j\frac{1}{2(1+\pi^2)}\right) e^{j\pi t} 1(t) + \left(\frac{1}{2\pi(1+\pi^2)} - j\frac{1}{2(1+\pi^2)}\right) e^{-j\pi t} 1(t). \end{aligned}$$

The two complex exponentials can be rewritten using a formula given in class:

$$(a + jb)e^{(c+jd)t} + (a - jb)e^{(c-jd)t} = |a + jb| e^{ct} \cos(dt + \angle(a + jb)), \quad (3)$$

where

$$|a + jb| = \sqrt{a^2 + b^2}, \quad \angle(a + jb) = \arctan \frac{b}{a}.$$

For the particular numbers we have in $x(t)$, formula (3) leads to:

$$\begin{aligned} \left(\frac{1}{2\pi(1 + \pi^2)} + j \frac{1}{2(1 + \pi^2)} \right) e^{j\pi t} + \left(\frac{1}{2\pi(1 + \pi^2)} - j \frac{1}{2(1 + \pi^2)} \right) e^{-j\pi t} = \\ \frac{1}{2\pi\sqrt{1 + \pi^2}} e^0 \cos(\pi t + \arctan \pi) = \frac{1}{2\pi\sqrt{1 + \pi^2}} \cos(\pi t + \arctan \pi) \end{aligned}$$

Summarizing, the expression for the solution $x(t)$ becomes:

$$x(t) = \left(\frac{\pi - 1}{\pi} \right) 1(t) + \left(\frac{\pi}{1 + \pi^2} \right) e^t 1(t) + \frac{1}{2\pi\sqrt{1 + \pi^2}} \cos(\pi t + \arctan \pi) 1(t).$$

7. (3 points) A signal $\{y_k\}$ is related to another signal $\{x_k\}$ by means of the formula:

$$y_k = \sum_{m=0}^k x_m, \quad k \geq 0.$$

If $\mathcal{Z}\{y_k\} = \frac{1}{(z-1)^2}$ and $x_0 = 0$, what are the numerical values of x_1 and x_2 ?

Solution: Observe that the formula above for y_k is in fact a convolution:

$$y_k = x_k * 1_k = \sum_{m=0}^k x_m 1_{k-m} = \sum_{m=0}^k x_m, \quad k \geq 0.$$

Therefore, by the properties of the \mathcal{Z} transform, we have that

$$\frac{1}{(z-1)^2} = X(z) \frac{z}{z-1} \quad \implies \quad X(z) = \frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}.$$

To take the inverse \mathcal{Z} transform, we multiply both sides of the last equation by z :

$$zX(z) = -1 + \frac{z}{z-1}.$$

In this way,

$$\mathcal{Z}^{-1}\{zX(z)\} = x_{k+1} = -\delta_k + 1_k, \quad k \geq 0.$$

From here,

$$x_1 = -\delta_0 + 1_0 = -1 + 1 = 0, \quad x_2 = -\delta_1 + 1_1 = 0 + 1 = 1.$$

Note: You need to have practice with the properties of the \mathcal{Z} transform and know how to obtain the \mathcal{Z} transforms of signals.

8. (10 points) Solve the following difference equation using the \mathcal{Z} transform method:

$$y_{k+2} + 7y_{k+1} + 10y_k = (-1)^k 1_k$$

with initial conditions $y_0 = 1$, $y_1 = -5$.

Solution: Denote by $Y(z) = \mathcal{Z}\{y_k\}$. Consider the general formula given in class:

$$\mathcal{Z}\{y_{k+n}\} = z^n Y(z) - \sum_{l=1}^n z^l y_{n-l}, \quad k \text{ is fixed and } n \geq 1.$$

Using this formula for $n = 1$ and $n = 2$ and substituting in the values for the initial conditions, we obtain:

$$\begin{aligned} \mathcal{Z}\{y_{k+1}\} &= zY(z) - zy_{1-1} = zY(z) - zy_0 = zY(z) - z, \\ \mathcal{Z}\{y_{k+2}\} &= z^2Y(z) - zy_{2-1} - z^2y_{2-2} = z^2Y(z) - zy_1 - z^2y_0 = z^2Y(z) + 5z - z^2. \end{aligned}$$

On the other hand, using the tables for the \mathcal{Z} transform,

$$\mathcal{Z}\{(-1)^k 1_k\} = \frac{z}{z+1}.$$

In all, the \mathcal{Z} transform of the difference equation becomes:

$$(z^2Y(z) + 5z - z^2) + 7(zY(z) - z) + 10Y(z) = \frac{z}{z+1},$$

or, equivalently,

$$(z^2 + 7z + 10)Y(z) = \frac{z}{z+1} + z^2 + 2z.$$

Solving for $Y(z)$ in this equation leads to:

$$Y(z) = \frac{z}{(z+1)(z^2+7z+10)} + \frac{z(z+2)}{z^2+7z+10}.$$

Now observe that $z^2 + 7z + 10 = (z+2)(z+5)$, which leads to the simplification:

$$Y(z) = \frac{z}{(z+1)(z+2)(z+5)} + \frac{z}{z+5} = zX(z) + \frac{z}{z+5}. \quad (4)$$

To obtain the inverse \mathcal{Z} transform of $Y(z)$ using the tables, we want to expand $Y(z)$ in fractions of the form:

$$Y(z) = \frac{Az}{z+5} + \frac{Bz}{z+1} + \frac{Cz}{z+2}.$$

We can use the partial fraction expansion method to do that. (*Note:* Before doing anything, notice that the partial fraction expansion method gives you expansions of the form:

$$Y(z) = \frac{A}{z+5} + \frac{B}{z+1} + \frac{C}{z+2},$$

since we want z multiplying in the numerators, we will use the partial fraction expansion method for $\frac{Y(z)}{z}$.) To save some computations, we will just expand the term $X(z)$ in the expression for $\frac{Y(z)}{z}$ (see equation (4)). Then we will substitute the result back in $Y(z)$.

Let $X(z)$ be the fraction:

$$X(z) = \frac{1}{(z+1)(z+2)(z+5)} = \frac{A}{z+5} + \frac{B}{z+1} + \frac{C}{z+2}.$$

Then we compute the constants A , B and C as follows:

$$\begin{aligned} A &= (z+5)X(z)|_{z=-5} = \frac{1}{(z+1)(z+2)} \Big|_{z=-5} = \frac{1}{(-4)(-3)} = \frac{1}{12}, \\ B &= (z+1)X(z)|_{z=-1} = \frac{1}{(z+5)(z+2)} \Big|_{z=-1} = \frac{1}{4}, \\ C &= (z+2)X(z)|_{z=-2} = \frac{1}{(z+5)(z+1)} \Big|_{z=-2} = \frac{1}{3(-1)} = -\frac{1}{3}. \end{aligned}$$

Substituting the expansion for $X(z)$ into $\frac{Y(z)}{z}$ we obtain:

$$\frac{Y(z)}{z} = \frac{1}{12(z+5)} + \frac{1}{4(z+1)} - \frac{1}{3(z+2)} + \frac{1}{z+5},$$

which implies that:

$$Y(z) = \left(\frac{1}{12} + 1 \right) \frac{z}{z+5} + \frac{z}{4(z+1)} - \frac{z}{3(z+2)}.$$

Using the tables, the inverse \mathcal{Z} transform of $Y(z)$ becomes:

$$y_k = \mathcal{Z}^{-1}\{Y(z)\} = \left(\frac{1}{12} + 1 \right) (-5)^k 1_k + \frac{1}{4} (-1)^k 1_k - \frac{1}{3} (-2)^k 1_k.$$

This gives the solution y_k for $k \geq 0$.

Note: To make sure you got the right solution, you can check if this equation satisfies the initial conditions. In fact,

$$1 = y_0 = \left(\frac{1}{12} + 1\right) (-5)^0 + \frac{1}{4}(-1)^0 - \frac{1}{3}(-2)^0 = \frac{1}{12} + 1 + \frac{1}{4} - \frac{1}{3} = 1,$$
$$-5 = y_1 = \left(\frac{1}{12} + 1\right) (-5) + \frac{1}{4}(-1) - \frac{1}{3}(-2) = \frac{-5}{12} - 5 - \frac{1}{4} + \frac{2}{3} = -5.$$

Therefore the y_k we have obtained satisfies the initial conditions.