Signals and Systems - MAE 143A

Final Exam - Winter Quarter 2007

Student name and number ______________________________

For all the questions you need to show ALL your work to get to the answer.

1. (4 points) For the time function

$$f(t) = \int_0^t e^{-3\tau}(t-\tau)e^{-2(t-\tau)}d\tau, \qquad t \geq 0\,.$$

   (i) (2 points) Find the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$.

   (ii) (2 points) Using $F(s)$ and the Final Value Theorem, obtain $\lim_{t\to\infty} f(t)$. Justify why you can use the Final Value Theorem.

Solution:

(i) Since $f(t)$ is a convolution, $f(t) = (e^{-3t}) * (te^{-2t})$, its Laplace transform can be computed as follows:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-3t}\} \cdot \mathcal{L}\{te^{-2t}\} = \frac{1}{s+3} \cdot \frac{1}{(s+2)^2}\,.$$

(ii) The Final Value Theorem can be applied for any function $F(s)$, such that the poles of $sF(s)$ are in the LHP, and it says that:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)\,.$$

In our case, the poles of $sF(s)$ can be obtained as:

$$(s+3)(s+2)^2 = 0 \quad \Longleftrightarrow \quad s = -3,\; s = -2 \text{ (double)}\,.$$

Since $-3, -2 < 0$ are in the LHP, then we can apply the Final Value Theorem. Therefore,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{(s+3)(s+2)^2} = 0\,.$$

2. (4 points) Given that

$$g(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad G(s) = \frac{s+1}{s(s+4)}\,,$$

find the Laplace transforms of the functions:

   (i) $g(2t)$,

(ii) $g(t-4)+g(t+2)$,

(iii) $tg(t)$,

(iv) $\mathrm{e}^{2t}g(t)$.

Solution:

(i) Using the time scaling property of Laplace transforms,

$$g(2t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{2}G\left(\frac{s}{2}\right) = \frac{\left(\frac{s}{2}\right)+1}{2\left(\frac{s}{2}\right)\left(\left(\frac{s}{2}\right)+4\right)} = \frac{s+2}{s(s+8)}\,.$$

(ii) Using the linearity and time shifting properties of Laplace transforms, we have that:

$$g(t-4)+g(t+2) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad \mathrm{e}^{-4s}G(s)+\mathrm{e}^{2s}G(s) = \left(\mathrm{e}^{-4s}+\mathrm{e}^{2s}\right)\frac{s+1}{s(s+4)}\,.$$

(iii) Using the property of multiplication by time, we have that:

$$tg(t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad -\frac{d}{ds}G(s) = -\frac{d}{ds}\left(\frac{s+1}{s(s+4)}\right) = \frac{s^2+2s+4}{s^2(s+4)^2}\,.$$

(iv) Using the frequency shift property, we have that:

$$\mathrm{e}^{2t}g(t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad G(s-2) = \frac{s-1}{(s-2)(s+2)}\,.$$

3. (4 points) Suppose that a system has a transfer function of the form:

$$P(s) = \frac{s+1}{(s-a)^2}\,,$$

for some $a \neq -1$. Knowing that we can expand $P(s)$ in fractions as:

$$P(s) = \frac{A}{s-a}+\frac{4}{(s-a)^2}\,,$$

(i) find the numerical values of A and a,

(ii) compute the *impulse* response to the system described by $P(s)$. Is the impulse response signal *stable*?

Solution:

(i) The numerical values are obtained according to Case 2 of the partial fraction expansion (PFE) method for repeated poles:

$$(s-a)^2 P(s)|_{s=a} = s+1|_{s=a} = a+1 = 4 \quad \Longrightarrow \quad a = 3\,,$$
$$\frac{d}{ds}\left((s-a)^2 P(s)\right)\Big|_{s=a} = 1 = A\,.$$

(ii) The impulse response to the system becomes:

$$p(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3} + \frac{4}{(s-3)^2}\right\} = \mathrm{e}^{3t}1(t) + 4t\mathrm{e}^{3t}1(t) = (1+4t)\mathrm{e}^{3t}1(t)\,.$$

The signal $p(t)$ is not stable because $\lim_{t\to\infty} p(t) = \infty$.

4. (6 points) Suppose a system is described by the transfer function:

$$P(s) = \frac{(s+1)^2}{s^2-1}\,.$$

Find the system response to the input $u(t) = \sin t$.

Solution: The response to the system is $y(t)$ such that $Y(s) = \mathcal{L}\{y(t)\}$ satisfies:

$$Y(s) = P(s)U(s)\,, \qquad U(s) = \mathcal{L}\{u(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}\,.$$

Therefore,

$$Y(s) = \frac{(s+1)^2}{(s^2-1)(s^2+1)} = \frac{s+1}{(s-1)(s+j)(s-j)}\,.$$

We expand $Y(s)$ in fractions,

$$Y(s) = \frac{c_1}{s-1} + \frac{c_2}{s-j} + \frac{c_3}{s+j}\,,$$

where the constants can be computed using Case 1 of the PFE method as follows:

$$c_1 = (s-1)Y(s)|_{s=1} = \frac{s+1}{s^2+1}\Big|_{s=1} = \frac{2}{2} = 1\,,$$
$$c_2 = (s-j)Y(s)|_{s=j} = \frac{s+1}{(s-1)(s+j)}\Big|_{s=j} = \frac{j+1}{2j(j-1)} = \frac{j+1}{-2-2j} = -\frac{(j+1)}{2(j+1)} = -\frac{1}{2}\,,$$
$$c_3 = \overline{c}_2 = -\frac{1}{2}\,.$$

Therefore,

$$Y(s) = \frac{1}{s-1} - \frac{1}{2(s-j)} - \frac{1}{2(s+j)} \quad \xrightarrow{\mathcal{L}^{-1}} \quad y(t) = \mathrm{e}^{t}1(t) - \frac{1}{2}\mathrm{e}^{jt}1(t) - \frac{1}{2}\mathrm{e}^{-jt}1(t)\,.$$

The complex exponentials can be put together using the formula $\cos t = \frac{1}{2}(\mathrm{e}^{jt} + \mathrm{e}^{-jt})$:

$$y(t) = (\mathrm{e}^{t} - \cos t)\cdot 1(t)$$

5. (6 points) A system is described by the ODE:

$$\ddot{x}(t) - 4\dot{x}(t) + 4x(t) = u(t)\,,$$

Using the Laplace transform method, find the solution to the system driven by the input $u(t) = \mathrm{e}^{-t}1(t)$ and with initial conditions $x(0) = 0$, $\dot{x}(0) = 4$.

Solution: The Laplace transform of the above ODE becomes:

$$\begin{aligned}(s^2X(s) - sx(0) - \dot{x}(0)) - 4(sX(s) - x(0)) + 4X(s) &= \frac{1}{s+1} \quad \Longleftrightarrow \\ (s^2 - 4s + 4)X(s) &= \frac{1 + (s-4)x(0)(s+1) + \dot{x}(0)(s+1)}{s+1} = \frac{5+4s}{s+1}\,.\end{aligned}$$

Solving for $X(s)$,

$$X(s) = \frac{5+4s}{(s^2-4s+4)(s+1)} = \frac{5+4s}{(s-2)^2(s+1)}\,.$$

We expand,

$$X(s) = \frac{c_1}{s-2} + \frac{c_2}{(s-2)^2} + \frac{c_3}{s+1}\,.$$

Using Case 2 of the PFE method for repeated poles, we get the value of the constants,

$$\begin{aligned}c_3 &= (s+1)X(s)|_{s=-1} = \left.\frac{5+4s}{s^1-4s+4}\right|_{s=-1} = \frac{1}{9}\,,\\ c_2 &= (s-2)^2X(s)\Big|_{s=2} = \left.\frac{5+4s}{s+1}\right|_{s=2} = \frac{5+8}{2+1} = \frac{13}{3}\,,\\ c_1 &= \frac{d}{ds}((s-2)^2X(s))\Big|_{s=2} = \frac{d}{ds}\left(\frac{5+4s}{s+1}\right)\Big|_{s=2} = \left.\frac{-1}{(s+1)^2}\right|_{s=2} = -\frac{1}{9}\,.\end{aligned}$$

From here,

$$X(s) = \frac{-1}{9(s-2)} + \frac{13}{3(s-2)^2} + \frac{1}{9(s+1)} \qquad \xrightarrow{\mathcal{L}^{-1}} \qquad x(t) = \left(-\frac{1}{9}\mathrm{e}^{2t} + \frac{13}{3}t\mathrm{e}^{2t} + \frac{1}{9}\mathrm{e}^{-t}\right)1(t)\,.$$

6. (6 points) A discrete time system has a transfer function

$$P(z) = \frac{z+3}{z^2 - \frac{1}{4}}\,.$$

(i) Find the difference equation governing the relationship between the input u_k and the output y_k, assuming zero initial conditions.

(ii) For the input $u_k = (-3)^k 1_{k-1}$, and $y_0 = 0$, $y_1 = 0$, find the output y_k.

Solution:

(i) Using the transfer function, the equation relating input $Y(z)$ and output $U(z)$ is:

$$Y(z) = \frac{z+3}{z^2 - \frac{1}{4}} U(z) \,. \tag{1}$$

This is equivalent to:

$$(z^2 - \frac{1}{4})Y(z) = (z+3)U(z) \quad \Longrightarrow \quad z^2 Y(z) - \frac{1}{4}Y(z) = zU(z) + 3U(z) \,.$$

The inverse $\mathcal{Z}$ transform of this equation becomes then:

$$y_{k+2} - \frac{1}{4} y_k = u_{k+1} + 3u_k \,, \qquad k \geq 0 \,.$$

(ii) Suppose now $u_k = (-3)^k 1_{k-1}$. Then,

$$\mathcal{Z}\{u_k\} = \mathcal{Z}\{(-3)^k 1_{k-1}\} = -3\mathcal{Z}\{(-3)^{k-1} 1_{k-1}\} = -3z^{-1}\frac{z}{z+3} = \frac{-3}{z+3} \,.$$

Plugging this into the input-output equation (1) we have that:

$$Y(z) = \frac{-3(z+3)}{(z^2 - \frac{1}{4})(z+3)} = \frac{-3}{z^2 - \frac{1}{4}} \,.$$

To find $\mathcal{Z}^{-1}\{Y(z)\}$, we rewrite this expression as:

$$z\frac{Y(z)}{z} = \frac{-3}{z^2 - \frac{1}{4}} = \frac{c_1}{z - \frac{1}{2}} + \frac{c_2}{z + \frac{1}{2}} \,.$$

The constants values can be obtained as:

$$c_1 = \left(z - \frac{1}{2}\right) Y(z)|_{z=1/2} = \left.\frac{-3}{z + \frac{1}{2}}\right|_{z=1/2} = \frac{-3}{\frac{1}{2} + \frac{1}{2}} = -3 \,,$$
$$c_2 = \left(z + \frac{1}{2}\right) Y(z)|_{z=-1/2} = \left.\frac{-3}{z - \frac{1}{2}}\right|_{z=-1/2} = \frac{-3}{-\frac{1}{2} - \frac{1}{2}} = 3 \,.$$

Therefore,

$$zY(z) = -\frac{3z}{z - \frac{1}{2}} + \frac{3}{z + \frac{1}{2}} \quad \xrightarrow{\mathcal{Z}^{-1}} \quad y_{k+1} = -3\left(\frac{1}{2}\right)^k 1_k + 3\left(-\frac{1}{2}\right)^k 1_k \,, \quad k \geq 0, \text{ and } y_0 = 0 \,.$$

7. (10 points) Solve the difference equation:

$$y_{k+2} + \frac{4}{3} y_{k+1} + \frac{1}{3} y_k = \left(\frac{1}{4}\right)^k \,, \quad k \geq 0 \,,$$

with initial conditions $y_0 = 1$ and $y_1 = -\frac{1}{3}$.

Solution: Taking the $\mathcal{Z}$ transform of both sides of the difference equation, we have that:

$$z^2 Y(z) - z^2 y_0 - z y_1 + \frac{4}{3} z Y(z) - \frac{4}{3} z y_0 + \frac{1}{3} Y(z) = \frac{z}{z - \frac{1}{4}} .$$

This implies:

$$(z^2 + \frac{4}{3} z + \frac{1}{3}) Y(z) = \frac{z}{z - \frac{1}{4}} + z^2 y_0 + \frac{4}{3} z y_0 + z y_1 .$$

Plugging in the initial conditions, this simplifies to:

$$(z^2 + \frac{4}{3} z + \frac{1}{3}) Y(z) = \frac{z}{z - \frac{1}{4}} + z^2 + \frac{4}{3} z - \frac{1}{3} z = \frac{z}{z - \frac{1}{4}} + z^2 + z ,$$

Solving now for $Y(z)$:

$$Y(z) = \frac{z}{(z - \frac{1}{4})(z^2 + \frac{4}{3} z + \frac{1}{3})} + \frac{(z^2 + z)}{z^2 + \frac{4}{3} z + \frac{1}{3}} .$$

Noticing that $z^2 + \frac{4}{3} z + \frac{1}{3}$ can be factored out as:

$$z^2 + \frac{4}{3} z + \frac{1}{3} = (z + 1)\left(z + \frac{1}{3}\right) ,$$

then the expression for $Y(z)$ simplifies as:

$$Y(z) = \frac{z}{(z - \frac{1}{4})(z + 1)(z + \frac{1}{3})} + \frac{z}{(z + \frac{1}{3})} .$$

Denote by $X(z)$ the fraction:

$$X(z) = \frac{1}{(z - \frac{1}{4})(z + 1)(z + \frac{1}{3})} ,$$

and expand it as (Case 1 of PFE method):

$$X(z) = \frac{c_1}{z - \frac{1}{4}} + \frac{c_2}{z + 1} + \frac{c_3}{z + \frac{1}{3}} .$$

The constants can be computed as:

$$c_1 = (z - \frac{1}{4}) X(z)|_{z=1/4} = \frac{1}{(z + 1)(z + \frac{1}{3})}\bigg|_{z=1/4} = \frac{48}{35} ,$$

$$c_2 = (z + 1) X(z)|_{z=-1} = \frac{1}{(z - \frac{1}{4})(z + \frac{1}{3})}\bigg|_{s=-1} = \frac{6}{5} ,$$

$$c_3 = (z + \frac{1}{3}) X(z)|_{z=-1/3} = \frac{1}{(z - \frac{1}{4})(z + 1)}\bigg|_{z=-1/3} = -\frac{18}{7} .$$

From here, we get an expanded expression for $Y(z)$:

$$Y(z) = \frac{48z}{35(z-\frac{1}{4})} + \frac{6z}{5(z+1)} + \frac{-18z}{7(z+\frac{1}{3})} + \frac{z}{(z+\frac{1}{3})} \quad \Longrightarrow$$

$$Y(z) = \frac{48z}{35(z-\frac{1}{4})} + \frac{6z}{5(z+1)} - \frac{11z}{7(z+\frac{1}{3})} \,.$$

The $\mathcal{Z}^{-1}$ of this function is then:

$$y_k = \mathcal{Z}^{-1}\{Y(z)\} = \frac{48}{35}\left(\frac{1}{4}\right)^k 1_k + \frac{6}{5}\,(-1)^k\,1_k - \frac{11}{7}\left(-\frac{1}{3}\right)^k 1_k \,.$$