Practice Questions for Final, 2009

The material for the Final will be based on all the homework (Homework 1 through Homework 4), the slides (signalsCT.pdf systemsCT.pdf, ImpulseResponse.pdf, systemsFD.pdf) and the text book (this is the material included in chapters 2, 4, 6, 8, and 12 of the book which is also common to the slides, see the website for more specific information.) The following are some additional examples for you to practice before the Final. Because there are already practice problems for Quiz 2, here we focus on questions more similar to Homework 4. Again, the problems in the Final can either be similar to the following exercises or to others like the ones in Homework 1 through Homework 4, Quiz 1, Quiz 2, and the practice problems for Quiz 2. Because of this, please study and practice all the homeworks, quizzes and practice-quiz solutions as well.

Before you start: Recall that for all the questions in the Quiz or exams you need to explain how you sustain your answer; otherwise you may lose some marks.

I. Questions on Transfer Function, Impulse Response, and Frequency Response. The objective of these problems is to determine either the system Transfer Function, its Impulse Response, or its Frequency Response. This is important because by knowing any of these, we can characterize the system response to any signal. Also, once you have determined one of them, you can easily find the others.

1. (10 points) A system is described by the following ODE:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 4y(t) = 7\frac{dx(t)}{dt}$$

Determine:

(i) The system Transfer Function.

(ii) The system Impulse Response.

Solution: (i) To obtain the Transfer Function, denote $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{x(t)\} = X(s)$. Then, by using the differentiation property of Laplace transforms, we transform the ODE into:

$$s^2Y(s) + 3sY(s) + 4Y(s) = 7sX(s).$$

From here, the Transfer Function of the system is:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{7s}{s^2 + 3s + 4}.$$

(ii) The system Impulse Response can be obtained by taking the inverse Laplace transform of $H(s)$. That is, we have that:

$$h(t) = \mathcal{L}^{-1}\left\{\frac{7s}{s^2 + 3s + 4}\right\}.$$

To compute the inverse Laplace transform, we will expand $H(s)$ into simple fractions, so that we can use the table of Laplace-transform pairs. To do this, first we find the roots r_1, r_2 of the denominator:

$$s^2+3s+4=0 \qquad \Longrightarrow \qquad s=\frac{-3\pm\sqrt{9-16}}{2}=\frac{-3\pm j\sqrt{7}}{2}.$$

For simplicity, we will denote the roots as $r_1=\dfrac{-3-j\sqrt{7}}{2}$, $r_2=\dfrac{-3+j\sqrt{7}}{2}$.

In other words, $s^2+3s+4=(s+\frac{3+j\sqrt{7}}{2})(s+\frac{3-j\sqrt{7}}{2})=(s-r_1)(s-r_2)$. Now, since the degree of the polynomial in the numerator is less than the degree in of the polynomial in the denominator, $H(s)$ admits a partial fraction expansion of the form:

$$H(s)=\frac{A}{s+\frac{3+j\sqrt{7}}{2}}+\frac{B}{s+\frac{3-j\sqrt{7}}{2}}=\frac{A}{s-r_1}+\frac{B}{s-r_2}.$$

To find the constants, several approaches can be followed. For example, one can sum the fractions above and then equate the numerators to find a system of equations in A and B (we have done this in class):

$$H(s)=\frac{7s}{s^2+3s+4}=\frac{A(s-r_2)+B(s-r_1)}{(s-r_1)(s-r_2)}=\frac{s(A+B)+(-Ar_2-Br_1)}{(s-r_1)(s-r_2)} \qquad \Longrightarrow$$

$$7s=s(A+B)+(A\frac{(3-j\sqrt{7})}{2}+B\frac{(3+j\sqrt{7})}{2}).$$

Now, equating equal powers of s in the polynomials on each side of the equation, we obtain:

$$s^1: \qquad 7=A+B \qquad \Longrightarrow \qquad 7r_2=Ar_2+Br_2$$

$$s^0: \qquad 0=A\frac{(3-j\sqrt{7})}{2}+B\frac{(3+j\sqrt{7})}{2}=A(-r_2)+B(-r_1).$$

Solving for A and B, we have that:

$$7r_2=B(-r_1+r_2)=B\left(\frac{(3+j\sqrt{7})}{2}-\frac{(3-j\sqrt{7})}{2}\right)=Bj\sqrt{7},$$

$$\Longrightarrow \qquad B=\frac{7r_2}{j\sqrt{7}}=j\sqrt{7}(-r_2)=j\sqrt{7}\left(\frac{3-j\sqrt{7}}{2}\right)=\frac{7+j3\sqrt{7}}{2}, \text{ and}$$

$$A=7-B=7-\frac{7+j3\sqrt{7}}{2}=\frac{7-j3\sqrt{7}}{2}$$

There are other approaches to find the constants A and B, see for example the exam solutions of other years, or the problems in the last section of this document. In the exam, you can use the method presented here or any other method that leads to the correct solution.

In all, we have that:

$$H(s)=\frac{A}{s-r_1}+\frac{B}{s-r_2}=\frac{7-j3\sqrt{7}}{2}\cdot\frac{1}{s+\frac{3+j\sqrt{7}}{2}}+\frac{7+j3\sqrt{7}}{2}\frac{1}{s+\frac{3-j\sqrt{7}}{2}}.$$

From here, we obtain the system impulse response as:

$$h(t) = \mathcal{L}^{-1}\left(\frac{7-j3\sqrt{7}}{2}\cdot\frac{1}{s+\frac{3+j\sqrt{7}}{2}}+\frac{7+j3\sqrt{7}}{2}\cdot\frac{1}{s+\frac{3-j\sqrt{7}}{2}}\right)$$

$$= \frac{7-j3\sqrt{7}}{2}\mathrm{e}^{-\left(\frac{3-j\sqrt{7}}{2}\right)t}u(t)+\frac{7+j3\sqrt{7}}{2}\mathrm{e}^{-\left(\frac{3+j\sqrt{7}}{2}\right)t}u(t).$$

2. (5 points) A system is described by the following ODE:

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+4y(t)=7\frac{dx(t)}{dt}$$

Determine:

(i) The system Frequency Response

(ii) Obtain the value of the magnitude and phase of the Frequency Response.

Solution:

(i) Observe that the ODE of this problem is the same as the ODE of Problem 1. We have computed already the Transfer Function for this system, $H(s)$. Then, the Frequency Response of the system is just $H(j\omega)$. That is, we have to substitute $s = j\omega$ in the expression of $H(s) = \dfrac{7s}{s^2+3s+4}$. In this way, we obtain:

$$H(j\omega) = \frac{j7\omega}{(j\omega)^2+3(j\omega)+4} = \frac{j7\omega}{4-\omega^2+j3\omega}\,.$$

(ii) In order to compute the magnitude of the Frequency Response and its phase, we simplify the expression above to put it into the form $H(j\omega) = a + jb$. To do this, we multiply by the complex conjugate of the denominator in $H(j\omega)$:

$$\begin{aligned} H(j\omega) &= \frac{j7\omega}{4-\omega^2+j3\omega}\cdot\frac{(4-\omega^2)-j3}{(4-\omega^2)-j3\omega} = \frac{j7\omega((4-\omega^2)-j3\omega)}{(4-\omega^2)^2+9\omega^2} \\ &= \frac{7\omega}{(4-\omega^2)^2+9\omega^2}\left(3\omega+j(4-\omega^2)\right) = \frac{7\cdot3\omega^2}{(4-\omega^2)^2+9\omega^2}+j\frac{7(4-\omega^2)\omega}{(4-\omega^2)^2+9\omega^2} \end{aligned}$$

From here, the magnitude of $H(j\omega)$ and its phase can be obtained as:

$$\begin{aligned} |H(j\omega)| &= \frac{7\omega}{(4-\omega^2)^2+9\omega^2}\sqrt{9\omega^2+(4-\omega^2)^2} = \frac{7\omega}{\sqrt{(4-\omega^2)^2+9\omega^2}}, \\ \angle H(j\omega) &= \arctan\left(\frac{\frac{7\omega}{(4-\omega^2)^2+9\omega^2}(4-\omega^2)}{\frac{7\omega}{(4-\omega^2)^2+9\omega^2}3\omega^2}\right) = \arctan\left(\frac{4-\omega^2}{3\omega^2}\right). \end{aligned}$$

II. System response to sinusoidal, co-sinusoidal, and complex-exponential inputs. If we know what is the Frequency Response of an LTI system, then it is immediate to compute its response to any sinusoid, co-sinusoid, or complex exponential. This is one of the key facts of this course and what motivates the introduction of the FD analysis of signals and systems.

1. (6 points) Suppose that the Frequency Response of an LTI system is given by:

$$H(j\omega) = \frac{1}{1+j\omega}.$$

Determine:

(i) The system response to $x_1(t) = 10\cos(\frac{\pi t}{2})$ as another cosinusoid.

(ii) The system response to $x_2(t) = 2\sin(\frac{t}{2} + \frac{\pi}{4})$ as another sinusoid.

(iii) The system response to $x_3(t) = \frac{1}{3}e^{j(-\frac{\pi}{3}t + \frac{\pi}{4})}$ as another complex exponential.

Solution: Before computing the system response to any of the signals given, we determine the Frequency Response magnitude and phase.

First, we rewrite $H(j\omega)$ in the form $H(j\omega) = a + jb$ as:

$$H(j\omega) = \frac{1}{1+j\omega} \cdot \frac{1-j\omega}{1-j\omega} = \frac{1-j\omega}{1+\omega^2} = \frac{1}{1+\omega^2} - \frac{j\omega}{1+\omega^2} = a + jb.$$

From here we have that:

$$|H(j\omega)| = \sqrt{a^2+b^2} = \frac{1}{1+\omega^2}\sqrt{1+\omega^2} = \frac{1}{\sqrt{1+\omega^2}},$$

$$\angle H(j\omega) = \arctan(\frac{b}{a}) = \arctan\left(\frac{-\omega/(1+\omega^2)}{1/(1+\omega^2)}\right) = \arctan\left(-\omega\right).$$

In fact, this system corresponds to a low-pass filter since $\lim_{\omega\to+\infty}|H(j\omega)| = 0$ and $|H(0)| = 1 \neq 0$. (This means that low-frequencies will be passed through the filter and high-frequencies will be attenuated.)

(i) The (cyclic) frequency, ω_0, of $x_1(t)$ is $\omega_0 = \frac{\pi}{2}$. Then, we will use the formula:

$$A\cos(\omega_0 t + \theta_0) \to A|H(j\omega_0)|\cos(\omega_0 t + \theta_0 + \angle H(j\omega_0))$$

In this way, the response to $x_1(t)$ is:

$$\begin{aligned} y_1(t) &= 10|H(j\frac{\pi}{2})|\cos(\frac{\pi t}{2} + \angle H(\frac{j\pi}{2})) \\ &= 10\frac{1}{\sqrt{1+\frac{\pi^2}{4}}}\cos\left(\frac{t\pi}{2} + \arctan\left(-\frac{\pi}{2}\right)\right) \\ &= \frac{20}{\sqrt{4+\pi^2}}\cos\left(\frac{t\pi}{2} + \arctan\left(-\frac{\pi}{2}\right)\right). \end{aligned}$$

Observe that this is the same result that we would obtain using the formula for complex exponentials:

$$Ae^{j\omega_0 t + \theta_0} \to A|H(j\omega_0)|e^{j(\omega_0 t + \theta_0 + \angle H(j\omega_0))}$$

and the formula

$$\cos x = \frac{\mathrm{e}^{jx} + \mathrm{e}^{-jx}}{2}.$$

In our case,

$10\cos(\frac{\pi t}{2}) = 10\frac{\mathrm{e}^{\frac{j\pi t}{2}} + \mathrm{e}^{-\frac{j\pi t}{2}}}{2}$, and then:

$$\mathrm{e}^{j\frac{\pi}{2}t} \to |H(j\frac{\pi}{2})|\mathrm{e}^{j(\frac{\pi}{2}t + \angle H(j\frac{\pi}{2}))} = \frac{1}{\sqrt{1+\frac{\pi}{4}}}\mathrm{e}^{j(\frac{\pi}{2}t + \arctan(-\frac{\pi}{2}))}$$

$$\mathrm{e}^{-j\frac{\pi}{2}t} \to |H(-j\frac{\pi}{2})|\mathrm{e}^{j(-\frac{\pi}{2}t + \angle H(-j\frac{\pi}{2})))} = \frac{1}{\sqrt{1+\frac{\pi}{4}}}\mathrm{e}^{-j(\frac{\pi}{2}t + \arctan(-\frac{\pi}{2}))}$$

therefore:

$$10\cos(\frac{\pi t}{2}) = 10\frac{\mathrm{e}^{\frac{j\pi t}{2}} + \mathrm{e}^{-\frac{j\pi t}{2}}}{2} \to \frac{10}{2\sqrt{1+\frac{\pi^2}{4}}}\left(\mathrm{e}^{-j(\frac{\pi}{2}t + \arctan(-\frac{\pi}{2}))} + \mathrm{e}^{-j(\frac{\pi}{2}t + \arctan(-\frac{\pi}{2}))}\right) =$$

$$= \frac{10}{\sqrt{1+\frac{\pi^2}{4}}}\cos\left(\frac{\pi}{2}t + \arctan(-\frac{\pi}{2})\right) = \frac{20}{\sqrt{4+\pi^2}}\cos\left(\frac{\pi}{2}t + \arctan(-\frac{\pi}{2})\right).$$

(ii) The frequency, ω_0, of $x_2(t)$ is $\omega_0 = \frac{1}{2}$. Then, the response to $x_2(t)$ is:

$$\begin{aligned} y_2(t) &= 2|H(\frac{j}{2})|\sin(\frac{t}{2} + \angle H(\frac{j}{2}) + \frac{\pi}{4}) \\ &= 2\frac{1}{\sqrt{1+\frac{1}{4}}}\sin\left(\frac{t}{2} + \arctan\left(-\frac{1}{2}\right) + \frac{\pi}{4}\right) \\ &= \frac{4}{\sqrt{5}}\sin\left(\frac{t}{2} + \arctan\left(-\frac{1}{2}\right) + \frac{\pi}{4}\right). \end{aligned}$$

(iii) The frequency, ω_3 of $x_3(t)$ is $\omega_3 = -\frac{\pi}{3}$. Then, the response to $x_3(t)$ is:

$$\begin{aligned} y_3(t) &= \frac{1}{3}|H(-\frac{j\pi}{3})|\exp\left(j(\frac{-t\pi}{3} + \angle H(\frac{-j\pi}{3}) + \frac{\pi}{4})\right) = \\ &= \frac{1}{3}\frac{1}{\sqrt{1+\frac{\pi^2}{9}}}\exp\left(j(\frac{-t\pi}{3} + \arctan\left(\frac{\pi}{3}\right) + \frac{\pi}{4})\right) \\ &= \frac{1}{\sqrt{9+\pi^2}}\exp\left(j(\frac{-t\pi}{3} + \arctan\left(\frac{\pi}{3}\right) + \frac{\pi}{4})\right). \end{aligned}$$

2. (5 points) An LTI system described by $H(j\omega)$ that is known to be a low-pass filter converts its input $x(t) = 2\sin(12t)$ into an output $y(t) = \sqrt{2}\sin(12t+\theta)$ for some real valued constant θ. Determine the value of the complex number $H(j12)$.

Solution: Given a sinusoid $x(t) = A\sin(\omega_0 t + \alpha_0)$, its output is

$$y(t) = A|H(j\omega_0)|\sin(\omega_0 t + \alpha_0 + \angle H(j\omega_0)).$$

From the expression for $x(t)$, we have that:

$$A = 2, \qquad \omega_0 = 12, \qquad \alpha_0 = 0\,.$$

Then, from the expression of $y(t)$ and the formula for the output of a sinusoid, it must be that:

$$2|H(j12)| = \sqrt{2}, \qquad \text{and} \qquad \angle H(j12) = \theta.$$

In this way, $H(j12) = \frac{1}{\sqrt{2}}e^{j\theta}$.

3. (3 points) A system converts its input

$$x(t) = 5\sin(12t)$$

into a steady-state output $y(t) = 25\sin(12t - \frac{\pi}{4}) + 2.5\sin(24t - \frac{\pi}{2})$. Is the system LTI?

Solution: No, the system can not be LTI, because the response of an LTI system to a sinusoid $x(t) = 5\sin(12t)$ would be:

$$\bar{y}(t) = 5|H(j12)|\sin(12t + \angle H(j12)).$$

The frequency component of $y(t)$ corresponding to $2.5\sin(24t - \frac{\pi}{2})$ would not be present. LTI systems do not create new frequencies that are not present in the inputs.

III. Questions on the Continuous Time Fourier Series (CTFS) expansions of periodic signals. Since the response of an LTI system to a sinusoid, co-sinusoid, or complex-exponential can be easily computed, then we can compute the response to any involved periodic signal. To do so, we first need to express the periodic signal as an infinite sum of complex exponentials or cosinusoids. This corresponds to finding the CTFS expansion of the signal.

This class of problems are aimed to test that you know how to compute these expansions. Other notions that are associated with a periodic signal are its average power and its spectrum. The spectrum of a signal is associated with the magnitude and phase of the harmonic function $X[k]$ of a signal $x(t)$. You should know how to plot these functions.

1. (7 points) Consider the signal $x(t) = |\sin(t)|$ with fundamental period $T_0 = \pi$. Find the complex form of the CTFS expansion for $x(t)$. Graph the magnitude and phase of the harmonic function $X[k]$ for $k = 0, \pm 1, \pm 2$.

Solution: The complex form of the CTFS expansion is given as:

$$x(t) = \sum_{k=-\infty}^{+\infty} X[k]\mathrm{e}^{j\omega_0 \pi t}, \qquad \text{where } \omega_0 = \frac{1}{T_0},$$

$$\text{and } X[k] = \frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{e}^{-j\omega_0 \pi t}dt\,.$$

First, we need to find the fundamental period of $x(t)$, T_0. The period of $\sin(t)$ is 2π, while the period of $|\sin(t)|$ is going to by $T_0 = \pi$ as it can be verified by looking at a graph of $|\sin(t)|$. In this way, $\omega_0 = \frac{2\pi}{\pi} = 2$.

Let us compute the harmonic function $X[k]$ using the formula above. Observe that $|\sin(t)| = \sin(t)$ for $0 \le t \le \pi$. Now, we can compute the integrals as follows:

$$\begin{aligned} X[k] &= \frac{1}{\pi}\int_0^{\pi} \sin(t)\mathrm{e}^{-jk2t}dt = \frac{1}{\pi}\int_0^{\pi}\left(\frac{\mathrm{e}^{jt}-\mathrm{e}^{-jt}}{2j}\right)\mathrm{e}^{-jk2t}dt = \\ &= \frac{1}{j2\pi}\int_0^{\pi}\left(\mathrm{e}^{jt-jk2t}-\mathrm{e}^{-jt-jk2t}\right)dt = \frac{1}{j2\pi}\left[\frac{\mathrm{e}^{jt-jk2t}}{j(1-k2)} - \frac{-\mathrm{e}^{-(jt+jk2t)}}{j(1+k2)}\right]_0^{\pi} = \\ &= -\frac{1}{2\pi}\left(\frac{\mathrm{e}^{(j-jk2)\pi}-1}{(1-k2)} + \frac{\mathrm{e}^{-(j+jk2)\pi}-1}{(1+k2)}\right) \end{aligned}$$

Now,

$$\begin{aligned} \mathrm{e}^{j(1-2k)\pi} &= \mathrm{e}^{j\pi}\mathrm{e}^{-2jk\pi} = (-1)\cdot 1 = -1, \\ \mathrm{e}^{-j(1+2k)\pi} &= \mathrm{e}^{-j\pi}\mathrm{e}^{-2jk\pi} = (-1)\cdot 1 = -1, \end{aligned}$$

for all possible integer values of k. (Recall the formula $\mathrm{e}^{j\alpha} = \cos\alpha + j\sin\alpha$, in this way $\mathrm{e}^{j\pi} = \mathrm{e}^{-j\pi} = -1$ and $\mathrm{e}^{-j2k\pi} = 1$.)

Therefore:

$$X[k] = \frac{1}{\pi}\left(\frac{1}{1-2k} + \frac{1}{1+2k}\right) = \frac{2}{\pi}\left(\frac{1}{1-4k^2}\right).$$

In this way, the CTFS of $x(t)$ becomes:

$$x(t) = \sum_{k=-\infty}^{+\infty} \frac{2}{\pi(1-4k^2)}\mathrm{e}^{jk2t}.$$

Now that we have found the harmonic function $X[k]$ of $x(t)$, its magnitude and phase are given by:

$$\begin{aligned} |X[k]| &= \left|\frac{2}{\pi(1-4k^2)}\right| = \frac{2}{\pi(1-4k^2)}, \\ \angle X[k] &= \arctan\left(\frac{0}{2/(\pi(1-4k^2)}\right) = 0\,. \end{aligned}$$

The graphs of the magnitude and phase of the harmonic function for the values of $k = 0, \pm 1, \pm 2$ are then:

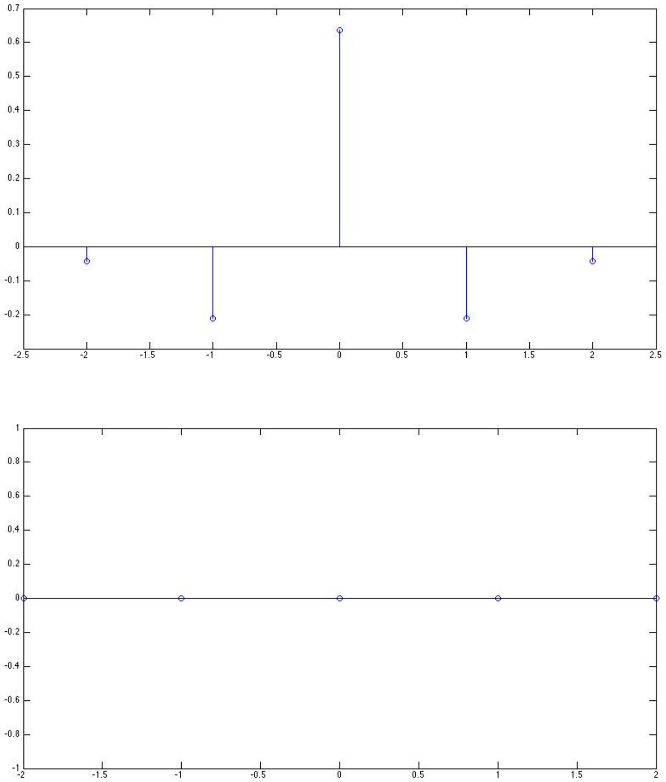

Figure 1: Magnitude and phase of $X[k]$ for Problem III.1.

2. (8 points) A periodic signal is given as:

$$x(t) = \mathrm{e}^{-\frac{t}{2}}, \qquad \text{for } 0 \leq t \leq 2,$$

over its fundamental period. Determine the exponential CTFS expansion for $x(t)$. Find the values of the magnitude and the phase of the harmonic function $X[k]$. After this, obtain the trigonometric form of the CTFS expansion.

Solution: Let us compute the harmonic function $X[k]$. Observe that, since $T_0 = 2$, then $\omega_0 = \frac{2\pi}{2} = \pi$. We have that:

$$X[k] = \frac{1}{2}\int_0^2 \mathrm{e}^{-\frac{t}{2}}\mathrm{e}^{-jk\pi t}dt = \frac{1}{2}\left[\frac{\mathrm{e}^{-(\frac{1}{2}+jk\pi)t}}{-(\frac{1}{2}+jk\pi)}\right]_0^2 = \frac{1-\mathrm{e}^{-(\frac{1}{2}+jk\pi)2}}{1+j2k\pi}$$

Using that $\mathrm{e}^{-j2\pi k} = \cos(-2\pi k) + j\sin(-2\pi k) = 1$, we have that

$$\begin{aligned} X[k] &= \frac{1-\mathrm{e}^{-(\frac{1}{2}+jk\pi)2}}{1+j2k\pi} = \frac{1-\mathrm{e}^{-1}}{1+j2k\pi} = \frac{1-\mathrm{e}^{-1}}{1+j2k\pi}\cdot\frac{1-j2k\pi}{1-j2k\pi} \\ &= \frac{(\mathrm{e}-1)}{\mathrm{e}(1+4k^2\pi^2)}(1-j2k\pi). \end{aligned}$$

Therefore, the CTFS of the signal $x(t)$ in complex form becomes:

$$x(t) = \sum_{k=-\infty}^{+\infty} \frac{(\mathrm{e}-1)(1-j2k\pi)}{\mathrm{e}(1+4k^2\pi^2)} \mathrm{e}^{jk2t}$$

The magnitude and phase of the harmonic function $X[k]$, becomes:

$$\begin{aligned}|X[k]| &= \frac{(\mathrm{e}-1)}{\mathrm{e}(1+4k^2\pi^2)}\sqrt{1+4k^2\pi^2} = \frac{\mathrm{e}-1}{\mathrm{e}\sqrt{1+4k^2\pi^2}}, \\ \angle X[k] &= \arctan\left(\frac{-2k\pi}{1}\right) = \arctan\left(-2k\pi\right), \qquad \text{for any integer } k \geq 0.\end{aligned}$$

Now, the trigonometric form of the CTFS is:

$$\begin{aligned}x(t) &= X[0] + \sum_{k=1}^{\infty} 2|X[k]|\cos(k\omega_0 t + \angle X[k]) = \\ &= \left(1-\frac{1}{\mathrm{e}}\right) + \sum_{k=1}^{\infty} 2\left(\frac{\mathrm{e}-1}{\mathrm{e}\sqrt{1+4k^2\pi^2}}\right)\cos(k\pi t + \arctan\left(-2k\pi\right)).\end{aligned}$$

IV. Questions on the system response to linear combinations of sinusoids, co-sinusoids, and complex-exponentials. Response to a CTFS expansion. Once we have a signal as a linear combination of sinusoids, cosinusoids, or a CTFS expansion we can compute the corresponding output to a LTI system. We just have to use the linearity property of LTI systems and what we have learnt from the set of problems in part II.

1. (12 points) Suppose that the Frequency Response of an LTI system is given as

$$H(j\omega) = \frac{1}{1+j\omega}.$$

Determine:

(i) The system response to $x_1(t) = 10\cos(\frac{\pi t}{2}) + \frac{1}{5}\sin(\frac{t}{2}+\pi)$.

(ii) The system response to $x_2(t) = \frac{1}{3}\mathrm{e}^{j(\frac{\pi}{4}t+\pi)} + 2\mathrm{e}^{j(\frac{\pi}{5}t)}$.

(iii) The system response to the signal $x_3(t) = \left(1-\frac{1}{\mathrm{e}}\right) + \sum_{k=1}^{\infty} 2\left(\frac{\mathrm{e}-1}{\mathrm{e}\sqrt{1+4k^2\pi^2}}\right)\cos(k\pi t + \arctan\left(-2k\pi\right))$

(iv) The system response to the signal $x_4(t) = \sum_{k=-\infty}^{+\infty} \frac{2}{\pi(1-4k^2)}\mathrm{e}^{jk2t}$.

Solution: Observe that this is the Frequency Response of the system in Problem II.1. So we start from the computations that were obtained there, on the magnitude and phase of the frequency response. That is, we have that:

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}},$$
$$\angle H(j\omega) = \arctan(-\omega).$$

To obtain the response of the LTI system, we use the fact that (a) we know how to compute the response to a sinusoid/cosinusoid/complex exponential, and (b)that the given input signals are a linear combination of these.

(i) The response of the system to $x_1(t)$, becomes:

$$\begin{aligned} y_1(t) &= 10|H(j\frac{\pi}{2})|\cos(\frac{\pi t}{2} + \angle H(j\frac{\pi}{2})) + \frac{1}{5}|H(j\frac{1}{2})|\sin(\frac{t}{2} + \pi + \angle H(j\frac{1}{2})) \\ &= \frac{20}{\sqrt{4+\pi^2}}\cos\left(\frac{t\pi}{2} + \arctan\left(-\frac{\pi}{2}\right)\right) + \frac{2}{5\sqrt{5}}\sin\left(\frac{t}{2} + \arctan\left(-\frac{1}{2}\right) + \pi\right). \end{aligned}$$

(ii) The response to $x_2(t)$ is:

$$\begin{aligned} y_2(t) &= \frac{1}{3}|H(j\frac{\pi}{4})|\mathrm{e}^{j(\frac{\pi}{4}t+\pi+\angle H(j\frac{\pi}{4}))} + 2|H(j\frac{\pi}{5})|\mathrm{e}^{j(\frac{\pi}{5}t+\angle H(j\frac{\pi}{5}))} = \\ &= \frac{4}{3\sqrt{16+\pi^2}}\mathrm{e}^{j(\frac{\pi}{4}t+\pi+\arctan\left(-\frac{\pi}{4}\right))} + \frac{10}{\sqrt{25+\pi^2}}\mathrm{e}^{j(\frac{\pi}{5}t+\arctan\left(-\frac{\pi}{5}\right))} \end{aligned}$$

(iii) The response to the signal $x_3(t)$ is the following:

$$\begin{aligned} y_3(t) &= |H(j\cdot 0)|\left(1-\frac{1}{e}\right) \\ &+ \sum_{k=1}^{+\infty} 2\left(\frac{e-1}{e\sqrt{1+4k^2\pi^2}}\right)|H(jk\pi)|\cos(k\pi t + \arctan(-2k\pi) + \angle H(jk\pi)) \\ &= \left(1-\frac{1}{e}\right) \\ &+ \sum_{k=1}^{+\infty} 2\left(\frac{e-1}{e\sqrt{1+4k^2\pi^2}}\frac{1}{\sqrt{1+k^2\pi^2}}\right)\cos(k\pi t + \arctan(-2k\pi) + \arctan(-k\pi)). \end{aligned}$$

(iv) The response to $x_4(t)$ is the following:

$$\begin{aligned} y_4(t) &= \sum_{k=-\infty}^{+\infty} |H(jk2)|\frac{2}{\pi(1-4k^2)}\mathrm{e}^{j(k2t+\angle H(jk2))} = \\ &= \sum_{k=-\infty}^{+\infty} \frac{2}{\pi(1-4k^2)\sqrt{1+4k^2\pi^2}}\mathrm{e}^{j(k2t+\arctan(-2k\pi))} \end{aligned}$$

2. (6 points) Consider the square-wave signal

$$x(t) = \sum_{n=1\ n \text{ is odd}} \frac{8}{n\pi} \cos\left(n\frac{\pi}{6}t - \frac{\pi}{2}\right)$$

with period $T = 12$. If $x(t)$ is the input of an LTI system with a frequency response

$$H(\omega) = \begin{cases} 1, & \text{for } |\omega| < 2 \text{ rad}/s, \\ 0, & \text{otherwise,} \end{cases}$$

determine the system output $y(t)$ and the average powers P_x and P_y of the input and output signals $x(t)$ and $y(t)$.

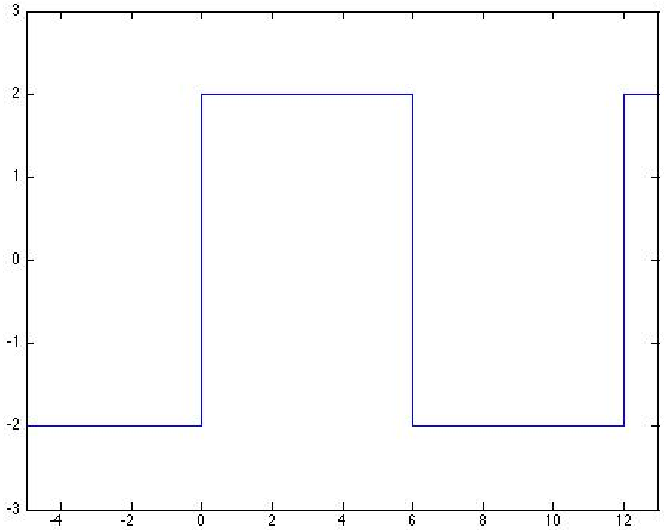

Figure 2: Square wave for Problem IV.2.

Solution: First, note that the input $x(t)$ consists of cosinusoids with harmonic frequencies

$$\frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \ldots, \frac{\text{rad}}{s}.$$

Since $H(\omega) = 0$ for $\omega > 2$, only the fundamental ($n = 1$) and the third harmonic ($n = 3$) of $x(t)$ will pass through the specified system. Hence,

$$y(t) = \frac{8}{\pi}\cos(\frac{\pi}{6}t - \frac{\pi}{2}) + \frac{8}{3\pi}\cos(3\frac{\pi}{6}t - \frac{\pi}{2}).$$

Note that Parseval formula was given for CTFS expansions in complex form:

$$y(t) = \sum_{k=-\infty}^{+\infty} Y[k]\mathrm{e}^{j\omega_0 tk} \quad \Longrightarrow \quad P_y = \sum_{k=-\infty}^{+\infty} |Y[k]|^2.$$

However the expansion that we have for $y(t)$ is in the trigonometric form. Recall that, when signals are real (like $y(t)$), the following equality holds:

$$y(t) = \sum_{k=-\infty}^{+\infty} Y[k] \mathrm{e}^{jk\omega_0 t} =$$

$$= Y[0] + \sum_{k=1}^{+\infty} 2|Y[k]| \cos(k\omega_0 t + \angle Y[k]) \tag{1}$$

where ω_0 is the fundamental frequency (see slides or book). In this way, from the output expression

$$y(t) = \frac{8}{\pi}\cos(\frac{\pi}{6}t - \frac{\pi}{2}) + \frac{8}{3\pi}\cos(3\frac{\pi}{6}t - \frac{\pi}{2})s \tag{2}$$

we can try to identify the coefficients $|Y[k]|$. We just have to match the coefficients of the cosines with the same frequency in formulas (1) and (2). Here, the fundamental frequency ω_0 we have to use in (1) is $\omega_0 = \frac{2\pi}{12} = \frac{\pi}{6}$, which is the fundamental frequency of $y(t)$ in (2).

Observe that in (2) we have a $\cos(\frac{\pi}{6}t - \frac{\pi}{2})$ with coefficient $\frac{8}{\pi}$. While in equation (1), there is only one cosine with frequency $\frac{\pi}{6}$: this is $\cos(\frac{\pi}{6}t + \angle Y[1])$ with coefficient $2|Y[1]|$. Therefore, $2|Y[1]| = \frac{8}{\pi}$ or $|Y[1]| = \frac{8}{2\pi}$.

On the other hand, the coefficient of $\cos(\frac{3\pi}{6}t - \frac{\pi}{2})$ in (2) is $\frac{8}{3\pi}$. And the coefficient of $\cos(\frac{3\pi}{6}t + \angle Y[3])$ in (1) is $2|Y[3]|$. In this way, $|Y[3]| = \frac{8}{6\pi}$. There are no more cosines in equation (2), so we don't need to match any more coefficients.

Now the formula for the average power becomes:

$$P_y = \left(\frac{8}{2\pi}\right)^2 + \left(\frac{8}{6\pi}\right)^2 \approx 1.0813.$$

On the other hand, the calculation of P_x is easier in the time domain, using the graph of $x(t)$. Note that the period is $T = 12$ and that $|x(t)|^2 = 4$ for $0 < t < 12$. Therefore,

$$P_x = \frac{1}{T}\int_0^T |x(t)|^2 dt = \frac{1}{12}\int_0^{12} 4 dt = 4.$$

V. Questions on the system response to any signal (not necessarily periodic) When the signal is not periodic, we can not use the CTFS expansion. Then, we need to use another approach, like the Fourier Transform or Laplace Transform. This set of problems are aimed to test that you know how to compute the Laplace transform of a signal (using the tables or basic Laplace Transform properties), how to compute an inverse Laplace transform, and how to compute the system response to any signal using Laplace transforms.

1. (7 points) Find the Laplace transform of the following function:

$$x(t) = \mathrm{e}^{7(t-1)}\mathrm{ramp}(t-1) * u(2t).$$

Solution: Denote by $x_1(t) = \mathrm{e}^{7(t-1)}\mathrm{ramp}(t-1)$ and its Laplace transform by $X_1(s)$. Denote by $x_2(t) = u(2t)$ and its Laplace transform by $X_2(s)$.
By the Convolution-Multiplication duality property, we have that:

$$\mathcal{L}\{x(t)\} = X_1(s)X_2(s),$$

so the problem reduces to finding $X_1(s)$ and $X_2(s)$.
To find $X_1(s)$ observe that $x_1(t) = x_3(t-1)$ with $x_3(t) = \mathrm{e}^{7t}\mathrm{ramp}(t)$. Thus, by the time-shift property we have that $X_1(s) = X_3(s)\mathrm{e}^{-s}$, where $X_3(s)$ is the Laplace transform of $x_3(t)$. This reduces the problem to finding $X_3(s)$ and $X_2(s)$.
Observe that the Laplace transform of $\mathrm{ramp}(t)$ can be easily found by using the tables. Moreover, $x_3(t) = \mathrm{e}^{7t}\mathrm{ramp}(t)$. Now by the complex-frequency shifting property, we have that:

$$\mathcal{L}\{\mathrm{e}^{7t}\mathrm{ramp}(t)\} = X_4(s-7), \quad \text{where } \mathcal{L}\{\mathrm{ramp}(t)\} = X_4(s) = \frac{1}{s^2}.$$

In this way, we have that:

$$\begin{aligned} X_4(s) &= \frac{1}{s^2}, \\ X_3(s) &= X_4(s-7) = \frac{1}{(s-7)^2}, \\ X_1(s) &= \mathrm{e}^{-s}X_3(s) = \frac{\mathrm{e}^{-s}}{(s-7)^2}. \end{aligned}$$

By looking up the table, we have that $\mathcal{L}\{u(t)\} = \frac{1}{s}$. Now, by the time-scaling property, we have that:

$$X_2(s) = \mathcal{L}\{u(2t)\} = \frac{1}{2}\frac{1}{\left(\frac{s}{2}\right)} = \frac{1}{s}.$$

Finally, putting everything together we obtain:

$$X(s) = \frac{\mathrm{e}^{-s}}{s(s-7)^2}.$$

2. (6 points) Suppose a system is described by the transfer function:

$$H(s) = \frac{(s+1)^2}{s^2-1}\,.$$

Find the system response to the input $x(t) = \sin t u(t)$.

Solution: The response to the system is $y(t)$ such that $Y(s) = \mathcal{L}\{y(t)\}$ satisfies:

$$Y(s) = H(s)X(s)\,, \qquad X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\sin t u(t)\} = \frac{1}{s^2+1}\,.$$

Therefore,

$$Y(s) = \frac{(s+1)^2}{(s^2-1)(s^2+1)} = \frac{s+1}{(s-1)(s+j)(s-j)}\,.$$

We expand $Y(s)$ in fractions,

$$Y(s) = \frac{c_1}{s-1} + \frac{c_2}{s-j} + \frac{c_3}{s+j}\,,$$

where the constants can be computed a partial fraction expansion method as follows:

$$\begin{aligned}
c_1 &= (s-1)Y(s)|_{s=1} = \left.\frac{s+1}{s^2+1}\right|_{s=1} = \frac{2}{2} = 1\,,\\
c_2 &= (s-j)Y(s)|_{s=j} = \left.\frac{s+1}{(s-1)(s+j)}\right|_{s=j} = \frac{j+1}{2j(j-1)} = \frac{j+1}{-2-2j} = -\frac{(j+1)}{2(j+1)} = -\frac{1}{2}\,,\\
c_3 &= \bar{c}_2 = -\frac{1}{2}\,.
\end{aligned}$$

Therefore,

$$Y(s) = \frac{1}{s-1} - \frac{1}{2(s-j)} - \frac{1}{2(s+j)} \quad \xrightarrow{\mathcal{L}^{-1}} \quad y(t) = \mathrm{e}^{t}u(t) - \frac{1}{2}\mathrm{e}^{jt}u(t) - \frac{1}{2}\mathrm{e}^{-jt}u(t)\,.$$

The complex exponentials can be put together using the formula $\cos t = \frac{1}{2}(\mathrm{e}^{jt} + \mathrm{e}^{-jt})$:

$$y(t) = (\mathrm{e}^{t} - \cos t)\cdot u(t)$$

3. (5 points) What is the expression of the signal $x_2(t)$ such that:

$$\mathcal{L}\{\mathrm{e}^{-t+2}u(t-5) * x_2(t)\} = \frac{\mathrm{e}^{-5s-3}}{(s+1)(s^2+1)}\,?$$

Explain what properties of the Laplace transform you use to get to the result.

Solution: We use the properties of the Laplace transform to determine $x_2(t)$. By the convolution-multiplication duality, we have that:

$$\mathcal{L}\{\mathrm{e}^{-t+2}u(t-5) * x_2(t)\} = \mathcal{L}\{\mathrm{e}^{-t+2}u(t-5)\}\mathcal{L}\{x_2(t)\} = \frac{\mathrm{e}^{-5s-3}}{(s+1)(s^2+1)}\,. \tag{3}$$

Let us compute now $\mathcal{L}\{\mathrm{e}^{-t+2}u(t-5)\}$. Adjusting the constant in the exponential and by the property of linearity of the Laplace transform, we have that:

$$\mathcal{L}\{\mathrm{e}^{-t+2}u(t-5)\} = \mathcal{L}\{\mathrm{e}^{-(t-5)-3}u(t-5)\} = \mathrm{e}^{-3}\mathcal{L}\{\mathrm{e}^{-(t-5)}u(t-5)\}\,.$$

By the time-shifting property of the Laplace transform, we have that:

$$\mathrm{e}^{-3}\mathcal{L}\{\mathrm{e}^{-(t-5)}u(t-5)\} = \mathrm{e}^{-3}\mathrm{e}^{-5s}\mathcal{L}\{\mathrm{e}^{-t}u(t)\} = \frac{\mathrm{e}^{-3-5s}}{s+1}\,.$$

Substituting this expression into (3), we obtain:

$$\frac{\mathrm{e}^{-3-5s}}{s+1}\mathcal{L}\{x_2(t)\} = \frac{\mathrm{e}^{-5s-3}}{(s+1)(s^2+1)} \quad \Longleftrightarrow \quad \mathcal{L}\{x_2(t)\} = \frac{s+1}{(s+1)(s^2+1)} = \frac{1}{s^2+1}$$

In this way,

$$x_2(t) = \mathcal{L}^{-1}\{\frac{1}{s^2+1}\} = \sin t \cdot u(t)\,.$$

4. (10 points) Suppose a system with transfer function $H_1(s) = \dfrac{1}{s^2+2s-3}$ is connected in series with another with trasfer function $H_2(s) = \dfrac{1}{s+2}$. What is the interconnected system impulse response? What is its transfer function? Is the system BIBO stable?

Solution: The cascaded system has a transfer function which is the product of the two transfer functions given. That is:

$$H(s) = H_1(s)H_2(s) = \frac{1}{(s^2+2s-3)(s+2)}\,.$$

Now the impulse response can be obtained as $h(s) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2-2s-3)(s+2)}\right\}$. Observe that:

$$s^2+2s-3 = (s-1)(s+3)\,.$$

In order to find the expression of $h(t)$, we first obtain the partial fraction expansion of $\frac{1}{(s-1)(s+3)(s+2)}$. Since there are no repeated poles, the PF expansion is of the form:

$$\frac{1}{(s-1)(s+3)(s+2)} = \frac{K_1}{s+3} + \frac{K_2}{(s+2)} + \frac{K_3}{s-1}\,.$$

The constants are determined by the following formulas:

$$\begin{aligned}
K_1 &= \frac{(s+3)}{(s-1)(s+3)(s+2)}\Big|_{(s=-3)} = \frac{1}{(s-1)(s+2)}\Big|_{(s=-3)} = \frac{1}{4}\,,\\
K_2 &= \frac{(s+2)}{(s-1)(s+3)(s+2)}\Big|_{(s=-2)} = \frac{1}{(s-1)(s+3)}\Big|_{(s=-2)} = \frac{-1}{3}\,,\\
K_3 &= \frac{s-1}{(s-1)(s+3)(s+2)}\Big|_{(s=1)} = \frac{1}{12}\,.
\end{aligned}$$

Therefore, we have that:

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{(s+3)} + \frac{-1}{3}\frac{1}{s+2} + \frac{1}{12}\frac{1}{s-1}\right\} = \frac{1}{4}\mathrm{e}^{-3t}\cdot u(t) - \frac{1}{3}\mathrm{e}^{-2t}\cdot u(t) + \frac{1}{12}\mathrm{e}^{t}\cdot u(t)\,.$$

The system is unstable because there is an exponential with positive real part, $\frac{1}{12}\mathrm{e}^t\cdot u(t)$, in the impulse response $h(t)$. This is going to make $\int_{-\infty}^{+\infty}|h(t)|dt = +\infty$.

5. (6 points) A system has a transfer function given by $H(s) = \dfrac{3s+1}{(s^2+3s+4)(s^2+2s)}$. Use the Final Value theorem to compute the final value of the impulse response. You have to justify why you can use this result.

Solution: The impulse response is $h(t)$ such that $\mathcal{L}\{h(t)\} = H(s)$. To compute the final value of $h(t)$ we can try to use the Final Value theorem.

To be able to apply the theorem, we need to guarantee that the poles of $sH(s)$ have negative real parts. In this case,

$$sH(s) = \frac{3s+1}{(s^2+3s+4)(s+2)}$$

with poles $p_1 = -2$ and $p_{2,3} = \frac{-3\pm\sqrt{9-16}}{2}$. These complex numbers satisfy that $Re(p_i) < 0$ for $i \in \{1,2,3\}$. Therefore, we can apply the Final Value theorem and obtain that:

$$\lim_{t\to+\infty} h(t) = \lim_{s\to 0} sH(s) = \lim_{s\to 0} \frac{3s+1}{(s^2+3s+4)(s+2)} = \frac{1}{8}\,.$$