

# Signals and Systems - MAE 143A

## Final Exam - Winter Quarter 2009

Student name and number \_\_\_\_\_

For all the questions you need to show ALL your work to get to the answer. Please justify your answers (a yes/no answer is not enough.)

**List of allowed materials:** Book, slides/class notes, homework, homework solutions

**List of not-allowed materials:** practice exercises for final, practice exercises for Quiz 2, previous exam solutions.

1. (9 points) The circuit of Figure 1 has as input the current  $i_S$  and as output the current through the inductor  $i_L$ . Here,  $R = 2\Omega$ , and  $L = 4H$ .
  - (i) Using Basic Principles, determine a first-order ODE model for the circuit that describes the relation between the input  $i_S$  and the output  $i_L$ .
  - (ii) What is a state for this system?
  - (iii) Determine the step response of the system.

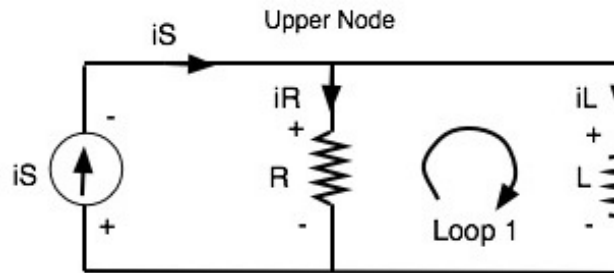


Figure 1: Circuit diagram for Problem 1

**Solution:** (i) Using KCL for the Upper Node, we obtain the equation:

$$i_S - i_R - i_L = 0 \quad \iff \quad i_S = i_R + i_L. \quad (1)$$

Using the KVL for Loop 1, we obtain:

$$V_R - V_L = 0 \quad \iff \quad V_R = V_L$$

Since the voltages are equal, let us use the same notation for them. That is,  $V = V_L = V_R$ .

The circuit-element laws give us:

$$\text{(Resistor)} \quad Ri_R = V \implies i_R = \frac{V}{2} \quad (2)$$

$$\text{(Inductor)} \quad L \frac{di_L}{dt} = V \implies 4 \frac{di_L}{dt} = V \implies 2 \frac{di_L}{dt} = \frac{V}{2} \quad (3)$$

Now, we substitute Equations (2), and (3) into Equation (1), and we obtain:

$$i_S = \frac{V}{2} + i_L = 2 \frac{di_L}{dt} + i_L.$$

Therefore, a first-order ODE modeling the input-output relation is:

$$2 \frac{di_L}{dt} + i_L = i_S, \quad \text{or, equivalently,} \quad \frac{di_L}{dt} + \frac{1}{2} i_L = \frac{1}{2} i_S$$

(ii) Since the model is given by a first-order ODE, a state for this system is trivially  $\mathbf{x} = i_L$ , which is the output of the system.

(iii) One possible approach to compute the step response is the following. We first compute the impulse response of the system:

$$h(t) = \mathcal{L}^{-1}(H(s)) = \mathcal{L}^{-1}\left(\frac{1/2}{s + 1/2}\right) = \frac{1}{2} e^{-\frac{1}{2}t} u(t).$$

Now the step response is just the integral of the impulse response:

$$\begin{aligned} y(t) &= \int_{-\infty}^t \frac{1}{2} e^{-\frac{1}{2}\tau} u(\tau) d\tau \\ &= \frac{1}{2} \int_0^t e^{-\frac{1}{2}\tau} d\tau = \left[-e^{-\frac{1}{2}\tau}\right]_0^t u(t) = \left[1 - e^{-\frac{1}{2}t}\right] u(t). \end{aligned}$$

2. (6 points) A system returns the output

$$y(t) = |x(t)|,$$

for a given input signal  $x(t)$ . Show that the system is neither homogeneous nor additive, but it is time-invariant and memoryless.

**Solution:**

Homogeneity:

Let  $x_1(t) = g(t)$ . Then  $y_1(t) = |g(t)|$ .

Let  $x_2(t) = Ag(t)$ . Then  $y_2(t) = |Ag(t)| = |A||g(t)| = |A|y_1(t)$ . In general we can not guarantee that  $|A||g(t)| = Ay_1(t)$  for any constant  $A$ . For example, if  $A = -5$ , we have that  $y_2(t) = 5y_1(t)$  and not  $y_2(t) = -5y_1(t)$ . Therefore, the system is not homogeneous.

Additivity:

Let  $x_1(t) = g_1(t)$ . Then  $y_1(t) = |g_1(t)|$ .

Let  $x_2(t) = g_2(t)$ . Then  $y_2(t) = |g_2(t)|$ .

Let  $x_3(t) = x_1(t) + x_2(t) = g_1(t) + g_2(t)$ . Then  $y_3(t) = |g_1(t) + g_2(t)|$ . In general we can not guarantee that  $|g_1(t) + g_2(t)| = |g_1(t)| + |g_2(t)| = y_1(t) + y_2(t)$  for any input signals  $g_1(t)$ ,  $g_2(t)$ . For example, take  $x_1(t) = 5$ , and  $x_2(t) = -5$ . Then  $x_3(t) = 0$  and  $y_3(t) = |0| = 0$ . However, we have that  $y_1(t) + y_2(t) = 5 + 5 = 10$ . In particular,  $10 = y_1(t) + y_2(t) \neq y_3(t) = 0$ . Therefore, the system is not additive.

Since the system is neither homogeneous nor additive, it is not linear.

Time invariance:

Let  $x_1(t) = g(t)$ . Then  $y_1(t) = |g(t)|$ .

Let  $x_2(t) = g(t - t_0)$ . Then  $y_2(t) = |g(t - t_0)|$ . This implies that  $y_2(t) = |g(t - t_0)| = y_1(t - t_0)$ .

Therefore, the system is time-invariant.

Memory:

The value of the output  $y(t) = |x(t)|$  only depends on the present value of the input  $x(t)$ , thus the system is memoryless.

3. (9 points) Using basic Laplace-transform pairs and the Laplace-transform properties determine the Laplace transform of the signal  $x(t) = -t \cos(\sqrt{2}t)u(t) * \frac{1}{2}e^{-2t}u\left(\frac{t}{2}\right)$ . What is the initial value of  $x(t)$ ? Justify your answer (if you are using certain Laplace transform property or theorem, mention it.) *Hint:* Observe that you can write  $e^{-2t} = e^{-4\frac{t}{2}}$ .

**Solution:** Let the Laplace transform of  $x(t)$  be  $X(s) = \mathcal{L}\{x(t)\}$ . The signal  $x(t)$  is given as the convolution of  $x_1(t) = -t \cos(\sqrt{2}t)u(t)$  and  $x_2(t) = e^{-2t}u\left(\frac{t}{2}\right)$ . The problem reduces to finding  $\mathcal{L}\{x_1(t)\} = X_1(s)$  and of  $\mathcal{L}\{x_2(t)\} = X_2(s)$  so that:

$$X(s) = X_1(s)X_2(s)$$

by the convolution-multiplication duality.

Consider first  $x_1(t) = -t \cos(\sqrt{2}t)u(t)$ . Denote by  $x_3(t) = \cos(\sqrt{2}t)u(t)$ . Then, using the tables we have that  $\mathcal{L}\{x_3(t)\} = X_3(s) = \frac{s}{s^2+2}$ . Now, using the property of complex-frequency differentiation, we have that:

$$-t \cos(\sqrt{2}t)u(t) = -t(\cos(\sqrt{2}t)u(t)) \xrightarrow{\mathcal{L}} \frac{d}{ds} \left( \frac{s}{s^2+2} \right) = \frac{s^2+2-2s^2}{(s^2+2)^2} = \frac{2-s^2}{(s^2+2)^2}.$$

Now consider  $x_2(t) = \frac{1}{2}e^{-2t}u\left(\frac{t}{2}\right)$ . In other words, we have that  $x_2(t) = \frac{1}{2}e^{-4\frac{t}{2}}u\left(\frac{t}{2}\right)$ . Consider the signal  $x_4(t) = e^{-4t}u(t)$ . We have that  $\mathcal{L}\{x_4(t)\} = \frac{1}{s+4}$ . Since  $x_2(t) = \frac{1}{2}x_4\left(\frac{t}{2}\right)$ , we can use the frequency scaling property to obtain:

$$X_2(s) = X_4(2s) = \frac{1}{2s+4}.$$

In this way we have that

$$X(s) = X_1(s)X_2(s) = \frac{2 - s^2}{(2s + 4)(s^2 + 2)^2}.$$

Now, to compute the initial value of  $x(t)$ , we will use the Initial Value Theorem, that says that:

$$\lim_{s \rightarrow 0} x(t) = \lim_{s \rightarrow \infty} \frac{s(2 - s^2)}{(2s + 4)(s^2 + 2)^2} = 0,$$

since the polynomial at the denominator has degree 5 and the polynomial at the numerator has degree  $3 < 5$ .

4. (10 points) Consider the interconnected system of Figure 2. Determine:
- The overall system impulse response,  $h(t)$ .
  - Whether the system is BIBO stable or not.

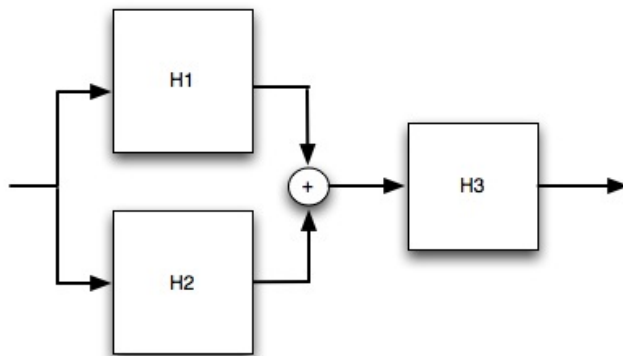


Figure 2: Block diagram for Problem 4. Here,  $H_1(s) = \frac{s}{s^2-4}$ ,  $H_2(s) = 1$ ,  $H_3(s) = \frac{1}{s+3}$ .

**Solution:** (i) We can obtain the impulse response of the system in Figure 2 by taking the inverse Laplace transform of the system transfer function.

By properties of interconnection, the system transfer function is

$$H(s) = (H_1(s) + H_2(s))H_3(s) = \left( \frac{s}{s^2-4} + 1 \right) \frac{1}{s+3} = \frac{s}{(s^2-4)(s+3)} + \frac{1}{s+3}.$$

We will decompose the transfer function into simpler fractions in order to take the inverse Laplace transform and use basic Laplace-transform pairs.

By expanding  $s^2 - 4 = (s - 2)(s + 2)$ , we look for an expansion of the form:

$$H(s) = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{1}{s+3}.$$

To compute the values of the constants, we equate:

$$\begin{aligned} \frac{s}{(s^2-4)(s+3)} &= \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s+3} \\ &= \frac{A(s+3)(s+2) + B(s-2)(s+3) + C(s^2-4)}{(s^2-4)(s+3)} \end{aligned}$$

Since numerators have to be equal, we have that:

$$s = A(s^2 + 5s + 6) + B(s^2 + s - 6) + C(s^2 - 4).$$

Equating coefficients of the same power of  $s^2$  above, gives:

$$\begin{aligned} s^2 : 0 &= A + B + C \\ s^1 : 1 &= 5A + B \\ s^0 : 0 &= 6A - 6B - 4C. \end{aligned}$$

From here it must be that  $B = 1 - 5A$ . Substituting into the first and third equation of the above system, we get:

$$\begin{aligned} 0 &= A + 1 - 5A + C = 1 - 4A + C, & \implies & 1 = 4A - C \\ 0 &= 6A - 6(1 - 5A) - 4C = -6 + 36A - 4C, & \implies & 6 = 36A - 4C. \end{aligned}$$

Then,  $C = 4A - 1$  and then  $6 = 36A - 4(4A - 1) = 36A - 16A + 4 = 20A + 4$ . That is,  $2 = 20A$  and  $A = 1/10$ . In this way,  $C = 4/10 - 1 = -6/10 = -3/5$  and  $B = 1 - 5/10 = 1/2$ . From here, we get to the partial fraction expansion:

$$H(s) = \frac{1}{10} \frac{1}{s-2} + \frac{1}{2} \frac{1}{s+2} - \frac{3}{5} \frac{1}{s+3} + \frac{1}{s+3}.$$

Now, it is easy to take the inverse Laplace transform of  $H(s)$ , which leads us to the final answer of part (i):

$$\begin{aligned} h(t) &= \frac{1}{10} e^{2t} u(t) + \frac{1}{2} e^{-2t} u(t) - \frac{3}{5} e^{-3t} u(t) + e^{-3t} u(t) = \\ &= \frac{1}{10} e^{2t} u(t) + \frac{1}{2} e^{-2t} u(t) + \frac{2}{5} e^{-3t} u(t). \end{aligned}$$

(ii) There is an exponential  $e^{2t} u(t)$  in the expression for the impulse response. Therefore we will have that  $\int_{-\infty}^{+\infty} |h(t)| dt = +\infty$  and then the system is not BIBO stable.

5. (8 points) A linear system has frequency response:

$$H(j\omega) = \frac{j\omega}{1 + j\omega}$$

Determine the system response to:

(i)  $x(t) = \sin(2t)$ ,

(ii)  $x(t) = \frac{1}{5} e^{j2t} + \frac{1}{2} e^{-j\pi t}$ .

**Solution:** (i) The response of the system to  $x(t) = \sin(2t)$  is another sinusoid, which is amplified and phase-shifted as follows:

$$y(t) = |H(j2)| \sin(2t + \angle(H(j2))).$$

In order to obtain  $y(t)$ , we then compute:

$$H(j2) = \frac{2j}{1+2j} = \frac{2j}{1+2j} \cdot \frac{1-2j}{1-2j} = \frac{2j+4}{5} = \frac{4}{5} + j\frac{2}{5}$$

$$|H(j2)| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \frac{\sqrt{20}}{5}$$

$$\angle H(j2) = \arctan\left(\frac{2/5}{4/5}\right) = \arctan\left(\frac{2}{4}\right) = \arctan\left(\frac{1}{2}\right).$$

Therefore, we have that:

$$y(t) = \frac{\sqrt{20}}{5} \sin\left(2t + \arctan\left(\frac{1}{2}\right)\right).$$

(ii) The system response to the signal  $x(t) = \frac{1}{5}e^{j2t} + \frac{1}{2}e^{-j\pi t}$  becomes:

$$y(t) = \frac{1}{5}|H(j2)|e^{j(2\pi t + \angle H(j2))} + \frac{1}{2}|H(-j\pi)|e^{-j(\pi t + \angle H(-j\pi))}.$$

We have already computed the magnitude and phase of  $H(j2)$ , let us compute the magnitude and phase of  $H(-j)$ :

$$H(-j\pi) = \frac{j(-j\pi)}{1+j(-j\pi)} = \frac{\pi}{1+\pi}$$

$$|H(-j\pi)| = \frac{\pi}{1+\pi},$$

$$\angle H(-j\pi) = \arctan\left(\frac{0}{\pi/(1+\pi)}\right) = 0.$$

Therefore, the system response becomes:

$$y(t) = \frac{\sqrt{20}}{25}e^{j(2t + \arctan(\frac{1}{2}))} + \frac{\pi}{2(1+\pi)}e^{-j\pi t}.$$

6. (13 points) A periodic signal  $x(t)$  is described as:

$$x(t) = \begin{cases} -1, & 0 \leq t \leq 2, \\ 0, & 2 < t \leq 4, \end{cases}$$

over its fundamental period  $T_0 = 4$ . The signal can be expressed as a Fourier series expansion in the form  $x(t) = \sum_{k=-\infty}^{+\infty} X[k]e^{jk\omega_0 t}$ . Graph the magnitude and phase of the harmonic function  $X[k]$  for the values  $k = 0, \pm 1, \pm 2$ . *Hint:* Recall that  $e^{jk\pi} = 1$  when  $k$  is even, and  $e^{jk\pi} = -1$  when  $k$  is odd.

**Solution:** (i) The CTFS in complex form is given by:

$$x(t) = \sum_{k=-\infty}^{+\infty} X[k]e^{jk\omega_0 t}, \quad \text{with } \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ and}$$

$$X[k] = \frac{1}{T_0} \int_0^{T_0} x(t)e^{-jk\omega_0 t} dt.$$

Using the above formula, we compute for  $k \neq 0$ :

$$\begin{aligned} X[k] &= \frac{1}{4} \int_0^4 x(t)e^{-jk\frac{2\pi}{4}t} dt = \frac{1}{4} \int_0^4 x(t)e^{-jk\frac{\pi}{2}t} dt \\ &= \frac{1}{4} \left( \int_0^2 x(t)e^{-jk\frac{\pi}{2}t} dt + \int_2^4 x(t)e^{-jk\frac{\pi}{2}t} dt \right) \\ &= \frac{1}{4} \left( \int_0^2 (-1) \cdot e^{-jk\frac{\pi}{2}t} dt + \int_2^4 0 \cdot e^{-jk\frac{\pi}{2}t} dt \right) \\ &= \frac{1}{4} \left( \left[ \frac{1}{jk\frac{\pi}{2}} e^{-jk\frac{\pi}{2}t} \right]_0^2 + 0 \right) \\ &= \frac{1}{jk2\pi} [e^{-jk\pi} - 1] = \frac{-j}{k2\pi} [e^{-jk\pi} - 1] = \begin{cases} -\frac{j}{4\ell\pi} [1 - 1] = 0 & \text{if } k \text{ is even, } k = 2\ell \neq 0 \\ -\frac{j}{4\ell\pi+2\pi} (-2) = \frac{j}{(2\ell+1)\pi} & \text{if } k \text{ is odd, } k = 2\ell + 1. \end{cases} \end{aligned}$$

The special case  $k = 0$  just simplifies to

$$\begin{aligned} X[k] &= \frac{1}{4} \int_0^4 x(t)e^{-jk\frac{2\pi}{4}t} dt = \frac{1}{4} \int_0^4 x(t)e^{-jk\frac{\pi}{2}t} dt \\ &= \frac{1}{4} \int_0^2 (-1) dt = \frac{1}{4} [-t]_0^2 = \frac{-2}{4} = -\frac{1}{2}. \end{aligned}$$

In other words, we have that:

$$\begin{aligned} X[0] &= -\frac{1}{2}, \\ X[2\ell] &= 0, \\ X[2\ell + 1] &= \frac{j}{(2\ell + 1)\pi}. \end{aligned}$$

Now, to be able to graph the magnitude and the phase of  $X[k]$  for the values of  $k = 0, \pm 1, \pm 2,$

we compute:

$$\begin{aligned} X[0] &= -\frac{1}{2}, & |X[0]| &= \frac{1}{2}, & \angle X[0] &= \pi, \\ X[\pm 2] &= 0, & |X[\pm 2]| &= 0, & \angle X[\pm 2] &= 0, \\ X[1] &= \frac{j}{\pi}, & |X[1]| &= \frac{1}{\pi}, & \angle X[1] &= \arctan\left(\frac{\frac{1}{\pi}}{0}\right) = +\frac{\pi}{2}. \\ X[-1] &= -\frac{j}{\pi}, & |X[-1]| &= \frac{1}{\pi}, & \angle X[-1] &= \arctan\left(\frac{\frac{-1}{\pi}}{0}\right) = -\frac{\pi}{2}. \end{aligned}$$