Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

# A CATALOG OF INVERSE-KINEMATICS PLANNERS FOR UNDERACTUATED SYSTEMS ON MATRIX GROUPS

Sonia Martínez and Jorge Cortés

Mechanical and Aerospace Engineering
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0411, USA

Francesco Bullo

Mechanical Engineering
University of California, Santa Barbara
2338 Engineering Building II
Santa Barbara, CA 9310-5070, USA

Abstract. This paper presents motion planning algorithms for underactuated systems evolving on rigid rotation and displacement groups. Motion planning is transcribed into (low-dimensional) combinatorial selection and inverse-kinematics problems. We present a catalog of solutions for all left-invariant underactuated systems on SE(2), SO(3), and SE(2) $\times \mathbb{R}$ classified according to their controllability properties.

1. **Introduction.** This paper presents motion planning algorithms for kinematic models of underactuated mechanical systems; i.e., systems with fewer actuators than degrees of freedom. From a practical perspective, the study of underactuated systems is motivated by performance, cost, and safety considerations. A fully actuated system requires more control inputs that an underactuated one, which requires more devices to generate the necessary forces. Moreover, underactuation provides a backup control technique for a failing fully actuated system.

We consider kinematic (i.e., driftless) models that are switched control systems, that is, dynamical systems described by a family of admissible vector fields and a control strategy that governs the switching between them. In particular, we focus on families of left-invariant vector fields defined on rigid displacements subgroups.

This class of systems arises in the context of kinematic modeling and kinematic reductions for mechanical control systems; see the example works [1, 4, 5, 6, 14, 15, 18], motivated by the following consideration. Previous work in robotics suggests that almost time-optimal trajectories for fully-actuated robots can be obtained by implementing a simple algorithmic decomposition. First, a path-planning problem in the configuration space (where obstacles and limitations are included) is solved; then a fast time-scaling algorithm over the given path can be applied; see [2].

2000 *Mathematics Subject Classification.* Primary: XXX, XXX; Secondary: XXX.

*Key words and phrases.* mechanical control systems, kinematics, motion planning.

The first author was partially supported by NSF award CAREER-CMMI-0643679. The second author was partially supported by NSF award CMMI-0908508. The third author was partially supported by NSF award CMS-0626457.

For systems that are kinematically controllable [6], this decomposition can also be performed without violating underactuation constraints. However, in this case, motions need to be restricted to integral curves of kinematic vector fields. The robot can switch among them by coming to rest first.

More detail about how the transcription into kinematic models simplifies the motion planning problem can be found in [6], which discusses 3R planar manipulators, and [3, 11] discussing the snakeboard system.

**Literature review.** Motion planning for kinematic models, sometimes referred to as driftless or nonholonomic models, is a classic problem in robotics; see [10] and also the references therein. In particular, the algorithms in [12, 13, 19] focus on dynamical aspects and exploit controllability properties.

On the other hand, the class of nonholonomic systems that motivate the work in this paper are special underactuated mechanical systems; see the definition in [4]. In particular, due to the technique of kinematic reductions outlined in the previous subsection (see also [4]), the search for a motion planning algorithm is closely related to the inverse-kinematics problem. Example inverse-kinematics methods include (i) iterative numerical methods for nonlinear optimization, see [9], (ii) geometric and decoupling methods for classes of manipulators, see [22, 24], (iii) the Paden-Kahan subproblems approach, see [19, 20], and (iv) the general polynomial programming approach, see [16]. The latter and more general method is based on tools from algebraic geometry and relies on simultaneously solving systems of algebraic equations.

Despite these efforts, no general methodology is currently available to solve these problems in closed-form. Accordingly, it is common to provide and catalog closed-form solutions for classes of relevant example systems; see [19, 22, 24].

Finally, it is worth mentioning that a number of papers study how to find state-dependent feedback transformations to render a given system into a form for which inverting controls can be found; see e.g., [8, 17, 21, 23], dealing with systems in chained form and control of unicycles. However, it is important to mention that these methods are not of use here since they would lead to trajectories for which the two-stage decoupling described earlier does not respect underactuation constraints.

**Problem statement.** We consider left-invariant control systems evolving on a matrix Lie subgroup $\mathrm{G} \subset \mathrm{SE}(3)$. Examples include systems on $\mathrm{SE}(2)$, $\mathrm{SO}(3)$ and $\mathrm{SE}(2) \times \mathbb{R}$. As usual in Lie group theory, we identify left-invariant vector fields with their value at the identity. Given a family of left-invariant vector fields $\{V_1, \dots, V_m\}$ on G, consider the associated driftless control system

$$\dot{g}(t) = \sum_{i=1}^{m} V_i(g(t)) u_i(t) \,, \tag{1}$$

where $g : \mathbb{R} \to \mathrm{G}$ and where the controls $(u_1, \dots, u_m)$ take value in $\{(\pm 1, 0, \dots, 0), (0, \pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1)\}$. For these systems, controllability can be assessed by algebraic means: it suffices to check the lack of involutivity of the distribution $\mathrm{span}\{V_1, \dots, V_m\}$. Recall that for matrix Lie algebras, Lie brackets are matrix commutators $[A, B] = AB - BA$.

This paper addresses the problem of how to compute feasible motion plans for the control system (1) by concatenating a finite number of flows along the input vector fields. We are interested in finding motion plans that employ a minimum number of flows—in terms of motion planning for mechanical systems using kinematic reductions this will imply that the robots will have to stop a minimum number of times.

We call a flow along any input vector field a *motion primitive* and its duration a *coasting time*. Therefore, motion planning is reduced to the problem of selecting a finite-length combination of $k$ motion primitives $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$ and computing appropriate coasting times $(t_1, \dots, t_k) \in \mathbb{R}^k$ that steer the system from the identity in the group to any target configuration $g_\mathrm{f} \in \mathrm{G}$. In mathematical terms, we need to solve

$$g_\mathrm{f} = \exp(t_1 V_{i_1}) \cdots \exp(t_k V_{i_k}).$$

Hence, motion planning is transcribed into low-dimensional combinatorial selection and inverse-kinematics problems.

**Contribution.** The contribution of this paper is a catalog of solutions for underactuated example systems defined on SE(2), SO(3), or SE(2) $\times \mathbb{R}$. Based on a controllability analysis, we classify families of underactuated systems that pose qualitatively different planning problems. For each family, we solve the planning problem for any target configuration by providing *a* combination of $k$ motion primitives and corresponding closed-form expressions for the coasting times. In each case, we attempt to select $k = \dim(\mathrm{G})$: generically, this is the minimum necessary (but sometimes not sufficient) number of motion primitives needed in order obtain a motion plan with a minimum number of stops. If the motion planning algorithm entails exactly $\dim(\mathrm{G})$ motion primitives, i.e., generically minimizes the number of switches, we will refer to it as a *switch-optimal* algorithm. Note that this does not mean that for particular isolated target configurations one cannot find a sequence of $\ell < \dim(\mathrm{G})$ motion primitives. Sections 2, 3, and 4 present switch-optimal planners for SE(2), SO(3), and SE(2) $\times \mathbb{R}$, respectively.

**Notation.** Here we briefly collect the notation used throughout the paper. Let $S$ be a set, $\mathrm{id}_S : S \to S$ denote the identity map on $S$ and let $\mathrm{ind}_S : \mathbb{R} \to \mathbb{R}$ denote the characteristic function of $S$, i.e., $\mathrm{ind}_S(x) = 1$ if $x \in S$ and $\mathrm{ind}_S(x) = 0$ if $x \notin S$. Let $\arctan2\,(x, y)$ denote the arctangent of $y/x$ taking into account which quadrant the point $(x, y)$ is in. We make the convention $\arctan2\,(0, 0) = 0$. Let $\mathrm{sign} : \mathbb{R} \to \mathbb{R}$ be the sign function, i.e., $\mathrm{sign}(x) = 1$ if $x > 0$, $\mathrm{sign}(x) = -1$ if $x < 0$ and $\mathrm{sign}(0) = 0$. Let $A_{ij}$ be the $(i, j)$ element of the matrix $A$. Given $v, w \in \mathbb{R}^n$, let $\arg(v, w) \in [0, \pi]$ denote the angle between them. Let $\| \cdot \|$ denote the Euclidean norm.

Given a family of left-invariant vector fields $\{V_1, \dots, V_m\}$ on G, we associate to each multiindex $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$ the forward-kinematics map $\mathcal{FK}^{(i_1,\dots,i_k)} : \mathbb{R}^k \to \mathrm{G}$ given by $(t_1, \dots, t_k) \mapsto \exp(t_1 V_{i_1}) \cdots \exp(t_k V_{i_k})$.

## 2. Catalog for SE(2).

Let $\{e_\theta, e_x, e_y\}$ be the basis of $\mathfrak{se}(2)$:

$$e_\theta = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; e_x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; e_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, $[e_\theta, e_x] = e_y$, $[e_y, e_\theta] = e_x$ and $[e_x, e_y] = 0$. For ease of presentation, we write $V \in \mathfrak{se}(2)$ as $V = a e_\theta + b e_x + c e_y \equiv (a, b, c)$, and $g \in \mathrm{SE}(2)$ as

$$g = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \equiv (\theta, x, y)\,.$$

With this notation, $\exp\colon \mathfrak{se}(2) \to \mathrm{SE}(2)$ is

$$\exp(a,b,c) = \left(a\,,\ \frac{\sin a}{a}b - \frac{1-\cos a}{a}c\,,\ \frac{1-\cos a}{a}b + \frac{\sin a}{a}c\right)$$

for $a \neq 0$, and $\exp(0,b,c) = (0,b,c)$.

**Lemma 2.1.** *(Controllability conditions). Consider two left-invariant vector fields $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ in $\mathfrak{se}(2)$. Their Lie closure is full rank if and only if $a_1b_2 - b_1a_2 \neq 0$ or $c_1a_2 - a_1c_2 \neq 0$.*

*Proof.* Given $[V_1, V_2] = (0, c_1a_2 - a_1c_2\,,\ a_1b_2 - b_1a_2)$, we have $\operatorname{span}\{V_1, V_2, [V_1, V_2]\} = \mathfrak{se}(2)$ if and only if

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & c_1a_2 - c_2a_1 & b_2a_1 - b_1a_2 \end{bmatrix} = (a_1b_2 - b_1a_2)^2 + (c_1a_2 - a_1c_2)^2 \neq 0\,.$$

□

Let $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ satisfy the controllability condition in Lemma 2.1. Accordingly, either $a_1$ or $a_2$ is different from zero. Without loss of generality, we will assume that $a_1 \neq 0$, and take $a_1 = 1$. As a consequence of Lemma 2.1, there are two qualitatively different cases to be considered:

$$\mathcal{S}_1 = \{(V_1, V_2) \in \mathfrak{se}(2) \times \mathfrak{se}(2) \mid V_1 = (1, b_1, c_1), V_2 = (0, b_2, c_2) \text{ and } b_2^2 + c_2^2 = 1\}\,,$$
$$\mathcal{S}_2 = \{(V_1, V_2) \in \mathfrak{se}(2) \times \mathfrak{se}(2) \mid V_1 = (1, b_1, c_1), V_2 = (1, b_2, c_2) \text{ and } (b_1 \neq b_2 \text{ or } c_1 \neq c_2)\}\,.$$

Since $\dim(\mathfrak{se}(2)) = 3$, we need at least three motion primitives along the flows of $\{V_1, V_2\}$ to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(1,2,1)}\colon \mathbb{R}^3 \to \mathrm{SE}(2)$. In the following propositions, we compute solutions for $\mathcal{S}_1$ and $\mathcal{S}_2$-systems.

**Proposition 1.** *(Inversion for $\mathcal{S}_1$-systems on $\mathrm{SE}(2)$). Let $(V_1, V_2) \in \mathcal{S}_1$. Consider the map $\mathcal{IK}[\mathcal{S}_1]\colon \mathrm{SE}(2) \to \mathbb{R}^3$,*

$$\mathcal{IK}[\mathcal{S}_1](\theta, x, y) = (\operatorname{arctan2}(\alpha, \beta)\,, \rho, \theta - \operatorname{arctan2}(\alpha, \beta)),$$

*where $\rho = \sqrt{\alpha^2 + \beta^2}$ and*

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2 \end{bmatrix} \left( \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} \right).$$

*Then, $\mathcal{IK}[\mathcal{S}_1]$ is a global right inverse of $\mathcal{FK}^{(1,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1)} \circ \mathcal{IK}[\mathcal{S}_1] = \mathrm{id}_{\mathrm{SE}(2)}\colon \mathrm{SE}(2) \to \mathrm{SE}(2)$.*

Note that the algorithm provided in the proposition is not only switch-optimal, but also works globally.

*Proof.* The proof follows from the expression of $\mathcal{FK}^{(1,2,1)}$. Let $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) = (\theta, x, y)$,

$$\theta = t_1 + t_3\,,$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} + \begin{bmatrix} b_2 & -c_2 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos t_1 \\ \sin t_1 \end{bmatrix} t_2\,.$$

The equation in $[x, y]^T$ can be rewritten as $[\alpha, \beta]^T = [\cos t_1, \sin t_1]^T t_2$. The selection $t_1 = \operatorname{arctan2}(\alpha, \beta)$, $t_2 = \rho$ solves this equation. □

**Proposition 2.** *(Inversion for $\mathcal{S}_2$-systems on* SE(2)*). Let $(V_1, V_2) \in \mathcal{S}_2$. Define the neighborhood of the identity in* SE(2)

$$U = \{(\theta, x, y) \in \mathrm{SE}(2) \mid \ \|(c_1 - c_2, b_1 - b_2)\|^2 \geq \max\{\|(x, y)\|^2, 2(1 - \cos\theta)\|(b_1, c_1)\|^2\}.$$

*Consider the map $\mathcal{IK}[\mathcal{S}_2] \colon U \subset \mathrm{SE}(2) \to \mathbb{R}^3$ whose components are*

$$\begin{aligned}
\mathcal{IK}[\mathcal{S}_2]_1(\theta, x, y) &= \mathrm{arctan2}\left(\rho, \sqrt{4 - \rho^2}\right) + \mathrm{arctan2}\left(\alpha, \beta\right), \\
\mathcal{IK}[\mathcal{S}_2]_2(\theta, x, y) &= \mathrm{arctan2}\left(2 - \rho^2, \rho\sqrt{4 - \rho^2}\right), \\
\mathcal{IK}[\mathcal{S}_2]_3(\theta, x, y) &= \theta - \mathcal{IK}[\mathcal{S}_2]_1(\theta, x, y) - \mathcal{IK}[\mathcal{S}_2]_2(\theta, x, y),
\end{aligned}$$

*and $\rho = \sqrt{\alpha^2 + \beta^2}$ and*

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\|(c_1 - c_2, b_1 - b_2)\|^2} \begin{bmatrix} c_1 - c_2 & b_2 - b_1 \\ b_1 - b_2 & c_1 - c_2 \end{bmatrix} \left( \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} \right).$$

*Then, $\mathcal{IK}[\mathcal{S}_2]$ is a local right inverse of $\mathcal{FK}^{(1,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1)} \circ \mathcal{IK}[\mathcal{S}_2] = \mathrm{id}_U \colon U \to U$.*

*Proof.* If $(\theta, x, y) \in U$, then

$$\rho = \|(\alpha, \beta)\| \leq \frac{1}{\|(c_1 - c_2, b_1 - b_2)\|} \cdot \left( \|(x, y)\| + \left\| \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} \right\| \right) \leq 2,$$

and hence $\mathcal{IK}[\mathcal{S}_2]$ is well-defined on $U$. Let $\mathcal{IK}[\mathcal{S}_2](\theta, x, y) = (t_1, t_2, t_3)$. The components of $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)$ are

$$\begin{aligned}
&\mathcal{FK}_1^{(1,2,1)}(t_1, t_2, t_3) = t_1 + t_2 + t_3, \\
&\begin{bmatrix} \mathcal{FK}_2^{(1,2,1)}(t_1, t_2, t_3) \\ \mathcal{FK}_3^{(1,2,1)}(t_1, t_2, t_3) \end{bmatrix} = \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} \\
&\qquad\qquad + \begin{bmatrix} c_1 - c_2 & b_1 - b_2 \\ b_2 - b_1 & c_1 - c_2 \end{bmatrix} \begin{bmatrix} \cos t_1 - \cos(t_1 + t_2) \\ \sin t_1 - \sin(t_1 + t_2) \end{bmatrix}.
\end{aligned}$$

In an analogous way to the previous proof, one verifies $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) = (\theta, x, y)$. □

**Remark 1.** The map $\mathcal{IK}[\mathcal{S}_2]$ in Proposition 2 is a local right inverse to $\mathcal{FK}^{(1,2,1)}$ on a domain that strictly contains $U$. In other words, our estimate of the domain of $\mathcal{IK}[\mathcal{S}_2]$ is conservative. For instance, for points of the form $(0, x, y) \in \mathrm{SE}(2)$, it suffices to ask for

$$\|(x, y)\| \leq 2\|(c_1 - c_2, b_1 - b_2)\|.$$

For a point $(\theta, 0, 0) \in \mathrm{SE}(2)$, it suffices to ask for

$$(1 - \cos\theta)\|(b_1, c_1)\|^2 \leq 2\|(c_1 - c_2, b_1 - b_2)\|^2.$$

Additionally, without loss of generality, it is convenient to assume that the vector fields $V_1$, $V_2$ satisfy $b_1^2 + c_1^2 \leq b_2^2 + c_2^2$, so as to maximize the domain $U$. On the other hand, note that given that the system is left-invariant, any target configuration $g_{\mathrm{f}} \in \mathrm{G}$ can be reached by the concatenation of an appropriately selected set of local inversions. •

We illustrate the performance of the algorithms in Fig. 1.

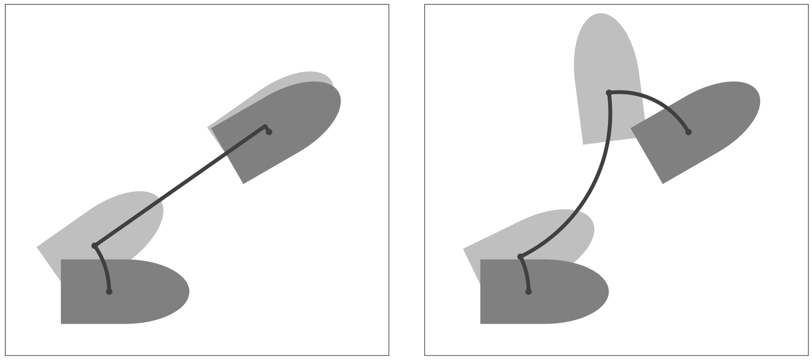

FIGURE 1. We illustrate the inverse-kinematics planners for $\mathcal{S}_1$ and $\mathcal{S}_2$-systems. For concreteness, one can think of the system as a vehicle moving in the plane. The parameters of both systems are $(b_1, c_1) = (0, .5)$, $(b_2, c_2) = (1, 0)$. The target location is $(\pi/6, 1, 1)$. Initial and target locations are depicted in dark gray.

3. **Catalog for SO(3).** Let $\{\widehat{e}_x, \widehat{e}_y, \widehat{e}_z\}$ be the basis of $\mathfrak{so}(3)$:

$$\widehat{e}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \ \widehat{e}_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \ \widehat{e}_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here we make use of the notation $\widehat{V} = a\widehat{e}_x + b\widehat{e}_y + c\widehat{e}_z \equiv \widehat{(a,b,c)}$ based on the Lie algebra isomorphism $\widehat{\cdot} : (\mathbb{R}^3, \times) \to (\mathfrak{so}(3), [\cdot,\cdot])$. Rodrigues formula [19] for the exponential $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$ is

$$\exp(\widehat{\eta}) = I_3 + \frac{\sin\|\eta\|}{\|\eta\|}\widehat{\eta} + \frac{1-\cos\|\eta\|}{\|\eta\|^2}\widehat{\eta}^2 .$$

The commutator relations are $[\widehat{e}_x, \widehat{e}_z] = -\widehat{e}_y$, $[\widehat{e}_y, \widehat{e}_z] = \widehat{e}_x$ and $[\widehat{e}_x, \widehat{e}_y] = \widehat{e}_z$.

**Lemma 3.1.** *(Controllability conditions). Consider two left-invariant vector fields $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ in $\mathfrak{so}(3)$. Their Lie closure is full rank if and only if $c_1a_2 - a_1c_2 \neq 0$ or $b_1c_2 - c_1b_2 \neq 0$ or $b_1a_2 - a_1b_2 \neq 0$.*

*Proof.* Given the equality $[\widehat{V}_1, \widehat{V}_2] = \widehat{V_1 \times V_2}$, with $V_1 \times V_2 = (b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1)$, one can see that $\operatorname{span}\{V_1, V_2, [V_1, V_2]\} = \mathfrak{so}(3)$ if and only if

$$\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ b_1c_2 - b_2c_1 & c_1a_2 - c_2a_1 & a_1b_2 - a_2b_1 \end{bmatrix} = (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2 \neq 0 .$$

□

Let $V_1$, $V_2$ satisfy the controllability condition in Lemma 3.1. Without loss of generality, we can assume $V_1 = e_z$ (otherwise we perform a suitable change of coordinates), and $\|V_2\| = 1$. In what follows, we let $V_2 = (a, b, c)$. Since $e_z$ and $V_2$ are linearly independent, necessarily $a^2 + b^2 \neq 0$ and $c \neq \pm 1$. Since $\dim(\mathfrak{so}(3)) = 3$, we need at least three motion primitives to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(1,2,1)}\colon \mathbb{R}^3 \to \mathrm{SO}(3)$, that is

$$\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) = \exp(t_1\widehat{e}_z)\exp(t_2\widehat{V}_2)\exp(t_3\widehat{e}_z)\,. \tag{2}$$

Observe that equation (2) is similar to the standard formula for Euler angles (see [19]) with the difference that here the rotation axes are not orthogonal. The following result is related to the problem of how to decompose a rotation by means of non-orthogonal Euler angles; see the papers [7, 25] for related results.

**Proposition 3.** *(Inversion for systems on* SO(3)*). Let* $V_1 = (0, 0, 1)$ *and* $V_2 = (a, b, c)$*, with* $a^2 + b^2 \neq 0$ *and* $c \neq \pm 1$*. Define the neighborhood of the identity in* SO(3)

$$U = \{R \in \mathrm{SO}(3) \mid\; R_{33} \in [2c^2 - 1, 1]\}.$$

*Consider the map* $\mathcal{IK}\colon U \subset \mathrm{SO}(3) \to \mathbb{R}^3$ *whose components are*

$$\begin{aligned}
\mathcal{IK}_1(R) &= \operatorname{arctan2}\left(w_1R_{13} + w_2R_{23}, -w_2R_{13} + w_1R_{23}\right)\,,\\
\mathcal{IK}_2(R) &= \arccos\left(\frac{R_{33} - c^2}{1 - c^2}\right)\,,\\
\mathcal{IK}_3(R) &= \operatorname{arctan2}\left(v_1R_{31} + v_2R_{32}, v_2R_{31} - v_1R_{32}\right)\,,
\end{aligned}$$

*where, for* $z = (1 - \cos(\mathcal{IK}_2(R)), \sin(\mathcal{IK}_2(R)))^T$,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} ac & b \\ cb & -a \end{bmatrix} z\,, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} ac & -b \\ cb & a \end{bmatrix} z.$$

*Then,* $\mathcal{IK}$ *is a local right inverse of* $\mathcal{FK}^{(1,2,1)}$*, that is, it satisfies* $\mathcal{FK}^{(1,2,1)} \circ \mathcal{IK} = \mathrm{id}_U : U \to U$*.*

*Proof.* Let $R \in U$. Then, $|\frac{R_{33}-c^2}{1-c^2}| \leq 1$, and hence $\mathcal{IK}(R)$ is well-defined. Denote $t_i = \mathcal{IK}_i(R)$ and let us show $R = \mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)$. Recall that the rows (resp. the columns) of a rotation matrix consist of orthonormal vectors in $\mathbb{R}^3$. Therefore, the matrix $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) \in \mathrm{SO}(3)$ is determined by its third column $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)e_z$ and its third row $e_z^T\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)$. The factors in (2) admit the following closed-form expressions. For $c_t = \cos t$ and $s_t = \sin t$,

$$\exp(t\widehat{e}_z) = \begin{bmatrix} c_t & -s_t & 0 \\ s_t & c_t & 0 \\ 0 & 0 & 1 \end{bmatrix}\,,$$

and $\exp(t\widehat{V}_2)$ equals

$$\begin{bmatrix} a^2 + (1 - a^2)c_t & ba(1 - c_t) - cs_t & ca(1 - c_t) + bs_t \\ ab(1 - c_t) + cs_t & b^2 + (1 - b^2)c_t & cb(1 - c_t) - as_t \\ ac(1 - c_t) - bs_t & bc(1 - c_t) + as_t & c^2 + (1 - c^2)c_t \end{bmatrix}\,.$$

Now, using the fact that $\exp(t\widehat{e}_z)e_z = e_z$, we get

$$\begin{aligned}
\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)e_z &= \exp(t_1\widehat{e}_z)\exp(t_2\widehat{V}_2)\exp(t_3\widehat{e}_z)e_z\\
&= \exp(t_1\widehat{e}_z)\exp(t_2\widehat{V}_2)e_z = \exp(t_1\widehat{e}_z)\begin{bmatrix} w_1 \\ w_2 \\ R_{33} \end{bmatrix} = Re_z\,.
\end{aligned}$$

A similar computation shows that $e_z^T\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) = e_z^TR$, which concludes the proof. □

**Remark 2.** If $\widehat{e}_z$ and $V_2$ are perpendicular, then $U = SO(3)$ and the map $\mathcal{IK}$ is a global right inverse of $\mathcal{FK}^{(1,2,1)}$. Otherwise, let us provide an equivalent formulation of the constraint $R_{33} \in [2c^2-1, 1]$ in terms of the axis/angle representation of the rotation matrix $R$. Recall that there always exist a, possibly non-unique, rotation angle $\theta \in [0, \pi]$ and an unit-length axis of rotation $\omega \in \mathbb{S}^2$ such that $R = \exp(\widehat{\omega}\theta)$. Because $\widehat{\omega}^2 = \omega^T\omega - I_3$, an equivalent statement of Rodrigues formula is

$$R = I_3 + \widehat{\omega}\sin\theta + (1-\cos\theta)(\omega^T\omega - I_3).$$

From $e_z^T\omega = \cos(\arg(e_z, \omega))$, we compute

$$\begin{aligned} e_z^T R e_z &= e_z^T e_z + (1-\cos\theta)((e_z^T\omega)^2 - e_z^T e_z) \\ &= 1 + (1-\cos\theta)((e_z^T\omega)^2 - 1) \\ &= 1 - \sin^2(\arg(e_z,\omega))(1-\cos\theta)\,. \end{aligned} \tag{3}$$

Therefore, $R_{33} \in [2c^2-1, 1]$ if and only if

$$1 - \sin^2(\arg(e_z,\omega))(1-\cos\theta) \geq 2c^2 - 1 \iff \sin^2(\arg(e_z,\omega))(1-\cos\theta) \leq 2(1-c^2)\,.$$

Two sufficient conditions are also meaningful. In terms of the rotation angle, if $|\theta| \leq \arccos(2c^2-1)$ then $1-\cos\theta \leq 2(1-c^2)$, and in turn equation (3) is satisfied. In terms of the axis of rotation, a sufficient condition for equation (3) is $\sin^2(\arg(e_z,\omega)) \leq \sin^2(\arg(e_z, V_2)) = 1 - c^2$. •

We illustrate the performance of the algorithm in Fig. 2.

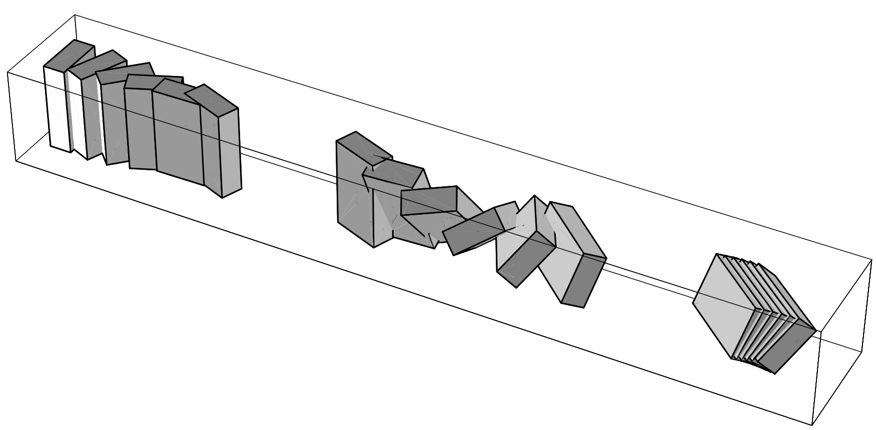

FIGURE 2. We illustrate the inverse-kinematics planner on SO(3). For concreteness, one can think of the system as a satellite rotating in space. The system parameters are $(a, b, c) = (0, 1/\sqrt{2}, 1/\sqrt{2})$. The target final rotation is $\exp(\pi/3, \pi/3, 0)$. To render the sequence of three rotations visible, the body is translated along the inertial $x$-axis.

4. **Catalog for** $\mathrm{SE}(2)\times\mathbb{R}$**.** Let $\{(e_\theta,0),(e_x,0),(e_y,0),(0,0,0,1)\}$ be a basis of $\mathfrak{se}(2)\times\mathbb{R}$, where $\{e_\theta,e_x,e_y\}$ stands for the basis of $\mathfrak{se}(2)$ introduced in Section 2. With a slight abuse of notation, we let $e_\theta$ denote $(e_\theta,0)$, and we similarly redefine $e_x$ and $e_y$. We also let $e_z=(0,0,0,1)$. The only non-vanishing Lie algebra commutators are $[e_\theta,e_x]=e_y$ and $[e_\theta,e_y]=-e_x$.

A left-invariant vector field $V$ in $\mathfrak{se}(2)\times\mathbb{R}$ is written as $V=ae_\theta+be_x+ce_y+de_z\equiv(a,b,c,d)$, and $g\in\mathrm{SE}(2)\times\mathbb{R}$ as $g=(\theta,x,y,z)$. The exponential map, $\exp:\mathfrak{se}(2)\times\mathbb{R}\longrightarrow\mathrm{SE}(2)\times\mathbb{R}$, is given component-wise by the exponential on $\mathfrak{se}(2)$ and $\mathbb{R}$, respectively. That is, $\exp(V)$ is equal to

$$\left(a\,,\ \frac{\sin a}{a}b-\frac{1-\cos a}{a}c\,,\ \frac{1-\cos a}{a}b+\frac{\sin a}{a}c\,,\ d\right)$$

if $a\neq 0$, and $\exp(V)=(0,b,c,d)$ if $a=0$.

**Lemma 4.1.** *(Controllability conditions for systems in* $\mathrm{SE}(2)\times\mathbb{R}$ *with 2 inputs). Consider two left-invariant vector fields* $V_1=(a_1,b_1,c_1,d_1)$ *and* $V_2=(a_2,b_2,c_2,d_2)$ *in* $\mathfrak{se}(2)\times\mathbb{R}$*. Their Lie closure is full rank if and only if* $a_2d_1-d_2a_1\neq 0$*, and either* $c_1a_2-a_1c_2\neq 0$ *or* $a_1b_2-b_1a_2\neq 0$*.*

*Proof.* Since $[V_1,V_2]=(0\,,c_1a_2-a_1c_2\,,\,a_1b_2-b_1a_2\,,0)\neq 0$, we deduce that either $c_1a_2-a_1c_2\neq 0$ or $a_1b_2-b_1a_2\neq 0$. In particular, this implies that necessarily $a_1\neq 0$ or $a_2\neq 0$. Assume $a_1\neq 0$. Now,

$$[V_1,[V_1,V_2]]=(0,a_1(-b_2a_1+b_1a_2),a_1(c_1a_2-c_2a_1),0)\,,$$

and note that $[V_2,[V_1,V_2]]=(a_2/a_1)[V_1,[V_1,V_2]]$. Higher-order Lie brackets yield nothing new. Finally, $\overline{\mathrm{Lie}}(\{V_1,V_2\})=\mathfrak{se}(2)\times\mathbb{R}$ if and only if

$$\det\begin{bmatrix} b_1 & c_1 & d_1 & a_1\\ b_2 & c_2 & d_2 & a_2\\ c_1a_2-c_2a_1 & b_2a_1-b_1a_2 & 0 & 0\\ a_1(-b_2a_1+b_1a_2) & a_1(c_1a_2-c_2a_1) & 0 & 0\end{bmatrix}$$
$$=a_1(a_2d_1-d_2a_1)\left[(c_1a_2-c_2a_1)^2+(-b_2a_1+b_1a_2)^2\right]\neq 0.$$

Since $[V_1,V_2]\neq 0$, this reduces to $a_2d_1-d_2a_1\neq 0$. □

Let $V_1$, $V_2$ satisfy the controllability condition in Lemma 4.1. Without loss of generality, we can assume $a_1=1$. As in the case of SE(2), there are two qualitatively different situations to be considered:

$$\mathcal{T}_1=\{(V_1,V_2)\in(\mathfrak{se}(2)\times\mathbb{R})^2\mid V_1=(1,b_1,c_1,d_1),\ V_2=(0,b_2,c_2,1)\text{ and }b_2^2+c_2^2\neq 0\}\,,$$
$$\mathcal{T}_2=\{(V_1,V_2)\in(\mathfrak{se}(2)\times\mathbb{R})^2\mid V_1=(1,b_1,c_1,d_1),\ V_2=(1,b_2,c_2,d_2),d_1\neq d_2\text{ and }(b_1\neq b_2\text{ or }c_1\neq c_2)\}\,.$$

**Lemma 4.2.** *(Controllability conditions for* $\mathrm{SE}(2)\times\mathbb{R}$ *systems with 3 inputs). Consider three left-invariant vector fields* $V_i=(a_i,b_i,c_i,d_i)$*,* $i=1,2,3$ *in* $\mathfrak{se}(2)\times\mathbb{R}$*. Assume* $\overline{\mathrm{Lie}}(\{V_{i_1},V_{i_2}\})\subsetneq\mathfrak{se}(2)\times\mathbb{R}$*, for* $i_j\in\{1,2,3\}$ *and* $\overline{\mathrm{Lie}}(\{V_1,V_2,V_3\})=\mathfrak{se}(2)\times\mathbb{R}$*. Then, possibly after a reordering of the vector fields, they must fall in one of the* *following cases:*

$$\begin{aligned}\mathcal{T}_3 = \{&(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \mid V_1 = (1, b_1, c_1, d_1),\\ &V_2 = (0, b_2, c_2, 0), V_3 = (1, b_1, c_1, d_3), d_1 \neq d_3 \text{ and } b_2^2 + c_2^2 \neq 0\},\\ \mathcal{T}_4 = \{&(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \mid V_1 = (1, b_1, c_1, d_1),\\ &V_2 = (0, b_2, c_2, 0), V_3 = (0, 0, 0, d_3), 0 \neq d_3 \neq d_1 \text{ and } b_2^2 + c_2^2 \neq 0\},\\ \mathcal{T}_5 = \{&(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \mid V_1 = (1, b_1, c_1, d_1),\\ &V_2 = (1, b_2, c_2, d_1), V_3 = (0, 0, 0, d_3), d_3 \neq 0 \text{ and } (b_2 \neq b_1 \text{ or } c_1 \neq c_2)\}.\end{aligned}$$

*Proof.* Without loss of generality, we can assume that $[V_1, V_2] \neq 0$ and $a_1 = 1$. Since $\overline{\text{Lie}}(\{V_1, V_2\}) \neq \mathfrak{se}(2) \times \mathbb{R}$, then $a_2 d_1 = d_2$. Given that the Lie closure of $\{V_1, V_2, V_3\}$ is full-rank, and $\dim(\text{span}\,\{V_1, V_2, [V_1, V_2]\}) = 3$, we have that $d_3 \neq a_3 d_1$. This latter fact, together with $\overline{\text{Lie}}(\{V_1, V_3\}) \subsetneq \mathfrak{se}(2) \times \mathbb{R}$, implies that $[V_1, V_3] = 0$, and therefore $b_3 = a_3 b_1$, $c_1 a_3 = c_3$. We now distinguish two situations depending on $[V_2, V_3]$ being zero or not.

*(a)* $[V_2, V_3] \neq 0$. Necessarily, $a_3 \neq 0$. Therefore, we can assume $a_3 = 1$. Since $\overline{\text{Lie}}(\{V_2, V_3\})$ is not full-rank, then $a_2 = 0$. We then have a $\mathcal{T}_3$-system.

*(b)* $[V_2, V_3] = 0$. Necessarily, $b_3 a_2 = b_2 a_3$ and $c_2 a_3 = c_3 a_2$. Depending on the values of $a_2$ and $a_3$, we consider:

- (i) If $a_2 = a_3 = 0$, then $d_2 = 0$, $d_3 \neq 0$, $b_3 = c_3 = 0$. Then, this is a $\mathcal{T}_4$-system.
- (ii) If $a_2 = 0$, and $a_3 = 1$, then $b_2 = b_3 a_2 = 0$, $c_2 = c_3 a_2 = 0$ and also $d_2 = d_1 a_2 = 0$. This is not possible as it would make $V_2 = 0$.
- (iii) If $a_2 = 1$ and $a_3 = 0$, then $b_3 = c_3 = 0$, and $d_2 = d_1$. Therefore, this is a $\mathcal{T}_5$-system.
- (iv) Finally, if $a_2 = 1$ and $a_3 = 1$, then $b_1 = b_2$, $c_1 = c_2$, and $d_1 = d_2$, which makes $V_1$ and $V_2$ linearly dependent.

□

4.1. **Two-dimensional input distribution.** Let $V_1$, $V_2$ satisfy the controllability condition in Lemma 4.1. Since $\dim(\mathfrak{se}(2) \times \mathbb{R}) = 4$, we need at least four motion primitives to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(2,1,2,1)} \colon \mathbb{R}^4 \to \text{SE}(2) \times \mathbb{R}$.

**Proposition 4.** *(Lack of switch-optimal inversion for $\mathcal{T}_1$-systems on* SE(2) × ℝ*). Let $(V_1, V_2) \in \mathcal{T}_1$. Then, the map $\mathcal{FK}^{(2,1,2,1)}$ in not invertible at any neighborhood of the origin.*

*Proof.* Let $\mathcal{FK}^{(2,1,2,1)}(t_1, t_2, t_3, t_4) = (\theta, x, y, z)$. Then,

$$\begin{aligned}\theta &= t_2 + t_4\,,\\ z &= t_1 + t_3 + d_1(t_2 + t_4) = t_1 + t_3 + d_1\theta\,,\end{aligned}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} c_1 & b_1 \\ -b_1 & c_1 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \begin{bmatrix} b_2 \\ c_2 \end{bmatrix} t_1 + \begin{bmatrix} b_2 & -c_2 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos t_2 \\ \sin t_2 \end{bmatrix} t_3\,.$$

Consider a configuration with $\theta = z = 0$. Then, the equation in $(x, y)$ is invertible if and only if the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\begin{bmatrix} t_2 \\ t_3 \end{bmatrix} \longmapsto \begin{bmatrix} \cos t_2 - 1 \\ \sin t_2 \end{bmatrix} t_3$$

is invertible. But $f$ can not be inverted in $(0, \beta)$, $\beta \neq 0$. □

**Remark 3.** An identical negative result holds if we start taking motion primitives along the flow of $V_1$ instead of $V_2$, i.e., if we consider the map $\mathcal{FK}^{(1,2,1,2)}\colon \mathbb{R}^4 \to \mathrm{SE}(2) \times \mathbb{R}$. •

Consider the map $\mathcal{FK}^{(1,2,1,2,1)}\colon \mathbb{R}^5 \to \mathrm{SE}(2) \times \mathbb{R}$.

**Proposition 5.** *(Inversion for $\mathcal{T}_1$-systems on $\mathrm{SE}(2)\times\mathbb{R}$). Let $(V_1, V_2) \in \mathcal{T}_1$. Consider the map $\mathcal{IK}[\mathcal{T}_1]\colon \mathrm{SE}(2) \times \mathbb{R} \to \mathbb{R}^5$ whose components are*

$$\begin{aligned}
\mathcal{IK}[\mathcal{T}_1]_1(\theta, x, y, z) &= \pi \mathrm{ind}_{]-\infty,0[}(\gamma - \rho) + \mathrm{arctan2}\,(\alpha, \beta) + \mathrm{arctan2}\,((\rho + \gamma)/2, 0)\,, \\
\mathcal{IK}[\mathcal{T}_1]_2(\theta, x, y, z) &= (\gamma - \rho)/2, \\
\mathcal{IK}[\mathcal{T}_1]_3(\theta, x, y, z) &= \mathrm{arctan2}\left((\rho^2 - \gamma^2)/4, 0\right) \\
&\quad + \pi\left(\mathrm{ind}_{]-\infty,0[}(\gamma + \rho) - \mathrm{ind}_{]-\infty,0[}(\gamma - \rho)\right), \\
\mathcal{IK}[\mathcal{T}_1]_4(\theta, x, y, z) &= (\gamma + \rho)/2, \\
\mathcal{IK}[\mathcal{T}_1]_5(\theta, x, y, z) &= \theta - \mathcal{IK}[\mathcal{T}_1]_1(\theta, x, y, z) - \mathcal{IK}[\mathcal{T}_1]_3(\theta, x, y, z),
\end{aligned}$$

*where $\rho = \sqrt{\alpha^2 + \beta^2}$ and*

$$\begin{aligned}
\gamma &= z - d_1\theta\,, \\
\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= \frac{1}{b_2^2 + c_2^2}\begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2\end{bmatrix}\left(\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix}\right).
\end{aligned}$$

*Then, $\mathcal{IK}[\mathcal{T}_1]$ is a global right inverse of $\mathcal{FK}^{(1,2,1,2,1)}$, i.e., it satisfies $\mathcal{FK}^{(1,2,1,2,1)} \circ \mathcal{IK}[\mathcal{T}_1] = \mathrm{id}_{\mathrm{SE}(2)\times\mathbb{R}}\colon \mathrm{SE}(2)\times\mathbb{R} \to \mathrm{SE}(2)\times\mathbb{R}$.*

*Proof.* The proof follows from the expression of $\mathcal{FK}^{(1,2,1,2,1)}$. First, let us denote $\mathcal{FK}^{(1,2,1,2,1)}(t_1, t_2, t_3, t_4, t_5) = (\theta, x, y, z)$,

$$\begin{aligned}
\theta &= t_1 + t_3 + t_5\,, \\
z &= t_2 + t_4 + d_1\theta\,, \\
\begin{bmatrix}x\\ y\end{bmatrix} &= \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix} + \begin{bmatrix}b_2 & -c_2\\ c_2 & b_2\end{bmatrix}\left(\begin{bmatrix}\cos t_1\\ \sin t_1\end{bmatrix} t_2 + \begin{bmatrix}\cos(t_1+t_3)\\ \sin(t_1+t_3)\end{bmatrix} t_4\right).
\end{aligned}$$

The equation in $[x, y]^T$ can be rewritten as

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\cos t_1\\ \sin t_1\end{bmatrix} t_2 + \begin{bmatrix}\cos(t_1+t_3)\\ \sin(t_1+t_3)\end{bmatrix} t_4\,,$$

which is solved by the selection of coasting times given by the components of the map $\mathcal{IK}[\mathcal{T}_1]$. □

**Proposition 6.** *(Inversion for $\mathcal{T}_2$-systems on $\mathrm{SE}(2)\times\mathbb{R}$). Let $(V_1, V_2) \in \mathcal{T}_2$. Define the neighborhood of the identity in $\mathrm{SE}(2)\times\mathbb{R}$*

$$\begin{aligned}
U = \Big\{&(\theta, x, y, z) \in \mathrm{SE}(2)\times\mathbb{R} \mid \\
&4\,\|(c_1 - c_2, b_1 - b_2)\|^2 \geq \max\{\|(x,y)\|^2\,, 2(1-\cos\theta)\|(b_1, c_1)\|^2\}\,, \\
&|z - d_1\theta| \leq 2|d_2 - d_1|\arccos\Big(-1 + \frac{\left(\|(x,y)\| + \|(b_1,c_1)\|\sqrt{2(1-\cos\theta)}\right)}{\|(c_1 - c_2, b_1 - b_2)\|}\Big)\Big\}.
\end{aligned}$$

*Consider the map* $\mathcal{IK}[\mathcal{T}_2]\colon \mathrm{SE}(2)\times\mathbb{R}\to\mathbb{R}^5$ *whose components are*

$$\begin{aligned}
\mathcal{IK}[\mathcal{T}_2]_1(\theta,x,y,z) &= \operatorname{arctan2}\left(l,\sqrt{4-l^2}\right)+\operatorname{arctan2}\left(\alpha,\beta\right),\\
\mathcal{IK}[\mathcal{T}_2]_2(\theta,x,y,z) &= 2\operatorname{arctan2}\left(\sqrt{4-l^2},l\right),\\
\mathcal{IK}[\mathcal{T}_2]_3(\theta,x,y,z) &= -\operatorname{arctan2}\left(\rho-l,\sqrt{4-(\rho-l)^2}\right)\\
&\qquad -\mathcal{IK}[\mathcal{T}_2]_1(\theta,x,y,z)-\mathcal{IK}[\mathcal{T}_2]_2(\theta,x,y,z),\\
\mathcal{IK}[\mathcal{T}_2]_4(\theta,x,y,z) &= \gamma-\mathcal{IK}[\mathcal{T}_2]_2(\theta,x,y,z),\\
\mathcal{IK}[\mathcal{T}_2]_5(\theta,x,y,z) &= \theta-\sum_{i=1}^{4}\mathcal{IK}[\mathcal{T}_2]_i(\theta,x,y,z),
\end{aligned}$$

*where* $\rho=\sqrt{\alpha^2+\beta^2}$*,* $s=\sin(\gamma/2)$*,* $c=\cos(\gamma/2)$ *and*

$$\begin{aligned}
\gamma &= (z-d_1\theta)/(d_2-d_1)\,,\\
l &= \frac{\rho(1+c)+\operatorname{sign}(\gamma)\sqrt{\rho^2(1+c)^2-(1+c)(2\rho^2-8s^2)}}{2(1+c)},\\
\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= \frac{1}{\|(d_1-d_2,c_1-c_2)\|^2}\begin{bmatrix}d_1-d_2 & c_2-c_1\\ c_1-c_2 & d_1-d_2\end{bmatrix}\left(\begin{bmatrix}x\\ y\end{bmatrix}-\begin{bmatrix}-d_1 & c_1\\ c_1 & d_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix}\right).
\end{aligned}$$

*Then,* $\mathcal{IK}[\mathcal{T}_2]$ *is a local right inverse of* $\mathcal{FK}^{(1,2,1,2,1)}$*, that is, it satisfies* $\mathcal{FK}^{(1,2,1,2,1)}\circ\mathcal{IK}[\mathcal{T}_2]=\mathrm{id}_U\colon U\to U$*.*

*Proof.* If $(\theta,x,y,z)\in U$, then $\rho\le 4$ and $|\gamma|\le 2\arccos(-1+\rho/2)$. This in turn implies that

$$c=\cos\left(\frac{\gamma}{2}\right)\ge -1+\frac{\rho}{2}\ge -1+\frac{\rho^2}{8}$$

over $\rho\le 4$. The second inequality guarantees that $l$ is well-defined. The first one implies $l\in[\rho-2,2]$, which makes $\mathcal{IK}[\mathcal{T}_2]$ well-defined on $U$. Let $\mathcal{IK}[\mathcal{T}_2](\theta,x,y,z)=(t_1,t_2,t_3,t_4,t_5)$. The components of $\mathcal{FK}^{(1,2,1,2,1)}(t_1,t_2,t_3,t_4,t_5)$ are the following

$$\begin{aligned}
\theta &= t_1+t_2+t_3+t_4+t_5\,,\\
z &= d_1\theta+(d_2-d_1)(t_2+t_4)\,,\\
\begin{bmatrix}x\\ y\end{bmatrix} &= \begin{bmatrix}-c_1\\ b_1\end{bmatrix}+\begin{bmatrix}c_1 & b_1\\ -b_1 & c_1\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}+\begin{bmatrix}c_1-c_2 & b_1-b_2\\ b_2-b_1 & c_1-c_2\end{bmatrix}\\
&\qquad\cdot\begin{bmatrix}\cos t_1-\cos(t_1+t_2)+\cos(t_1+t_2+t_3)-\cos(\sum_{i=1}^4 t_i)\\ \sin t_1-\sin(t_1+t_2)+\sin(t_1+t_2+t_3)-\sin(\sum_{i=1}^4 t_i)\end{bmatrix}
\end{aligned}$$

After some involved computations, one can verify $\mathcal{FK}^{(1,2,1,2,1)}(t_1,t_2,t_3,t_4,t_5)=(\theta,x,y,z)$. □

4.2. **Three-dimensional input distribution.** Let $V_1$, $V_2$, $V_3$ satisfy the controllability condition in Lemma 4.2. Consider $\mathcal{FK}^{(1,3,2,1)}\colon\mathbb{R}^4\to\mathrm{SE}(2)\times\mathbb{R}$.

**Proposition 7.** *(Inversion for $\mathcal{T}_3$-systems on* $\mathrm{SE}(2)\times\mathbb{R}$*). Let* $(V_1, V_2, V_3) \in \mathcal{T}_3$*. Consider the map* $\mathcal{IK}[\mathcal{T}_3]\colon \mathrm{SE}(2)\times\mathbb{R} \to \mathbb{R}^4$ *whose components are*

$$\begin{aligned}
\mathcal{IK}[\mathcal{T}_3]_1(\theta, x, y, z) &= \operatorname{arctan2}(\alpha, \beta) - \mathcal{IK}[\mathcal{T}_3]_2(\theta, x, y, z)\,,\\
\mathcal{IK}[\mathcal{T}_3]_2(\theta, x, y, z) &= \frac{z - d_1\theta}{d_3 - d_1},\\
\mathcal{IK}[\mathcal{T}_3]_3(\theta, x, y, z) &= \rho\,,\\
\mathcal{IK}[\mathcal{T}_3]_4(\theta, x, y, z) &= \theta - \operatorname{arctan2}(\alpha, \beta)\;,
\end{aligned}$$

*where* $\rho = \sqrt{\alpha^2 + \beta^2}$ *and*

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{b_2^2 + c_2^2}\begin{bmatrix} b_2 & c_2\\ -c_2 & b_2\end{bmatrix}\left(\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix}\right).$$

*Then,* $\mathcal{IK}[\mathcal{T}_3]$ *is a global right inverse of* $\mathcal{FK}^{(1,3,2,1)}$*, that is, it satisfies* $\mathcal{FK}^{(1,3,2,1)} \circ \mathcal{IK}[\mathcal{T}_3] = \mathrm{id}_{\mathrm{SE}(2)\times\mathbb{R}}\colon \mathrm{SE}(2)\times\mathbb{R} \to \mathrm{SE}(2)\times\mathbb{R}$*.*

*Proof.* The proof follows from the expression of $\mathcal{FK}^{(1,3,2,1)}$. first, if we let denote $\mathcal{FK}^{(1,3,2,1)}(t_1, t_2, t_3, t_4) = (\theta, x, y, z)$, then

$$\begin{aligned}
\theta &= t_1 + t_2 + t_4\,,\\
z &= d_1t_1 + d_3t_2 + d_1t_4 = d_1\theta + (d_3 - d_1)t_2\,,\\
\begin{bmatrix}x\\ y\end{bmatrix} &= \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix} + \begin{bmatrix}b_2 & -c_2\\ c_2 & b_2\end{bmatrix}\begin{bmatrix}\cos(t_1+t_2)\\ \sin(t_1+t_2)\end{bmatrix} t_3\,.
\end{aligned}$$

The equation in $[x, y]^T$ can be rewritten as

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\cos(t_1+t_2)\\ \sin(t_1+t_2)\end{bmatrix} t_3\,,$$

which is solved by the selection given by $(t_1, t_2, t_3, t_4) = \mathcal{IK}[\mathcal{T}_3](\theta, x, y, z)$. □

Consider the map $\mathcal{FK}^{(1,2,1,3)}\colon \mathbb{R}^4 \to \mathrm{SE}(2)\times\mathbb{R}$.

**Proposition 8.** *(Inversion for $\mathcal{T}_4$-systems on* $\mathrm{SE}(2)\times\mathbb{R}$*). Let* $(V_1, V_2, V_3) \in \mathcal{T}_4$*. Consider the map* $\mathcal{IK}[\mathcal{T}_4]\colon \mathrm{SE}(2)\times\mathbb{R} \to \mathbb{R}^4$ *given by*

$$\mathcal{IK}[\mathcal{T}_4](\theta, x, y, z) = \Big(\operatorname{arctan2}(\alpha, \beta)\,, \rho, \theta - \operatorname{arctan2}(\alpha, \beta)\,, \frac{z - d_1\theta}{d_3}\Big),$$

*where* $\rho = \sqrt{\alpha^2 + \beta^2}$ *and*

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{b_2^2 + c_2^2}\begin{bmatrix} b_2 & c_2\\ -c_2 & d_2\end{bmatrix}\left(\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix}\right).$$

*Then,* $\mathcal{IK}[\mathcal{T}_4]$ *is a global right inverse of* $\mathcal{FK}^{(1,2,1,3)}$*, that is, it satisfies* $\mathcal{FK}^{(1,2,1,3)} \circ \mathcal{IK}[\mathcal{T}_4] = \mathrm{id}_{\mathrm{SE}(2)\times\mathbb{R}}\colon \mathrm{SE}(2)\times\mathbb{R} \to \mathrm{SE}(2)\times\mathbb{R}$*.*

*Proof.* If $\mathcal{FK}^{(1,2,1,3)}(t_1, t_2, t_3, t_4) = (\theta, x, y, z)$, then

$$\begin{aligned}
\theta &= t_1 + t_3\,,\\
\begin{bmatrix}x\\ y\end{bmatrix} &= \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix} + \begin{bmatrix}b_2 & -c_2\\ c_2 & b_2\end{bmatrix}\begin{bmatrix}\cos t_1\\ \sin t_1\end{bmatrix} t_2\,,\\
z &= d_1(t_1 + t_3) + d_3t_4\,.
\end{aligned}$$

The equation in $[x, y]^T$ can be rewritten as $[\alpha, \beta]^T = [\cos t_1, \sin t_1]^T t_2$. As in the proof of Proposition 1, the selection $t_1 = \operatorname{arctan2}(\alpha, \beta)$, $t_2 = \rho$ solves it. □

**Proposition 9.** *(Inversion for $\mathcal{T}_5$-systems on* $\mathrm{SE}(2)\times\mathbb{R}$*). Let* $(V_1, V_2, V_3) \in \mathcal{T}_5$*. Define the neighborhood of the identity in* $\mathrm{SE}(2)\times\mathbb{R}$

$$U = \{(\theta, x, y) \in \mathrm{SE}(2)\times\mathbb{R} \mid \ \|(c_1 - c_2, b_1 - b_2)\|^2 \geq \max\{\|(x, y)\|^2, 2(1-\cos\theta)\|(b_1, c_1)\|^2\}.$$

*Consider the map* $\mathcal{IK}[\mathcal{T}_5]\colon U \subset \mathrm{SE}(2)\times\mathbb{R} \to \mathbb{R}^4$ *whose components are*

$$\begin{aligned}
\mathcal{IK}[\mathcal{T}_5]_1(\theta, x, y, z) &= \mathrm{arctan2}\left(\rho, \sqrt{4-\rho^2}\right) + \mathrm{arctan2}\left(\alpha, \beta\right),\\
\mathcal{IK}[\mathcal{T}_5]_2(\theta, x, y, z) &= \mathrm{arctan2}\left(2-\rho^2, \rho\sqrt{4-\rho^2}\right),\\
\mathcal{IK}[\mathcal{T}_5]_3(\theta, x, y, z) &= \theta - \mathcal{IK}[\mathcal{T}_5]_1(\theta, x, y) - \mathcal{IK}[\mathcal{T}_5]_2(\theta, x, y),\\
\mathcal{IK}[\mathcal{T}_5]_4(\theta, x, y, z) &= \frac{z - d_1\theta}{d_3},
\end{aligned}$$

*and* $\rho = \sqrt{\alpha^2+\beta^2}$ *and*

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{\|(c_1-c_2, b_1-b_2)\|^2}\begin{bmatrix}c_1-c_2 & b_2-b_1\\ b_1-b_2 & c_1-c_2\end{bmatrix}\left(\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix}\right).$$

*Then,* $\mathcal{IK}[\mathcal{T}_5]$ *is a local right inverse of* $\mathcal{FK}^{(1,2,1,3)}$*, that is, it satisfies* $\mathcal{FK}^{(1,2,1,3)} \circ \mathcal{IK}[\mathcal{T}_5] = \mathrm{id}_U\colon U \to U$*.*

*Proof.* If $(\theta, x, y, z) \in U$, then one can see that $\rho = \|(\alpha, \beta)\| \leq 2$, and therefore $\mathcal{IK}[\mathcal{T}_5]$ is well-defined on $U$. Let $\mathcal{IK}[\mathcal{T}_5](\theta, x, y, z) = (t_1, t_2, t_3, t_4)$. The components of $\mathcal{FK}^{(1,2,1,3)}(t_1, t_2, t_3, t_4)$ are

$$\begin{aligned}
\mathcal{FK}_1^{(1,2,1,3)}(t_1, t_2, t_3, t_4) &= t_1 + t_2 + t_3,\\
\begin{bmatrix}\mathcal{FK}_2^{(1,2,1,3)}(t_1, t_2, t_3, t_4)\\ \mathcal{FK}_3^{(1,2,1,3)}(t_1, t_2, t_3, t_4)\end{bmatrix} &= \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix}\\
&\quad + \begin{bmatrix}c_1-c_2 & b_1-b_2\\ b_2-b_1 & c_1-c_2\end{bmatrix}\begin{bmatrix}\cos t_1 - \cos(t_1+t_2)\\ \sin t_1 - \sin(t_1+t_2)\end{bmatrix},\\
\mathcal{FK}_4^{(1,2,1,3)}(t_1, t_2, t_3, t_4) &= d_1(t_1+t_2+t_3) + d_3 t_4.
\end{aligned}$$

One can verify that $\mathcal{FK}^{(1,2,1,3)}(t_1, t_2, t_3, t_4) = (\theta, x, y, z)$. □

We illustrate the performance of the algorithms in Fig. 3.

5. **Conclusions.** We have presented a catalog of feasible motion planning algorithms for underactuated controllable systems on SE(2), SO(3), and $\mathrm{SE}(2)\times\mathbb{R}$. Table 1 presents a summary of our results. Future directions of research include (i) considering other relevant classes of underactuated systems on SE(3), (ii) computing catalogs of optimal sequences of motion primitives, and (iii) developing hybrid feedback schemes that rely on the proposed open-loop planners to achieve point stabilization and trajectory tracking.

**Acknowledgments.** The authors would like to thank Professor Kevin M. Lynch for insightful discussions and the anonymous reviewers for useful remarks.

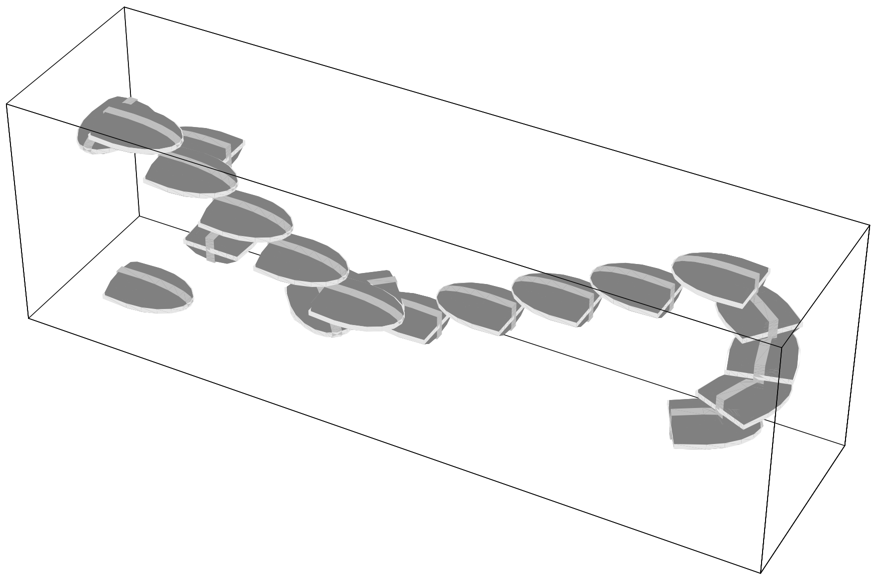

FIGURE 3. We illustrate the inverse-kinematics planner for a $\mathcal{T}_1$-system on $\mathrm{SE}(2)\times\mathbb{R}$. For concreteness, one can think of the system as a planar vehicle with vertical dynamics. The system parameters are $b_1 = 1$, $c_1 = 0$, $d_1 = .5$, $b_2 = -2$, and $c_2 = 0$. The target location is $(\pi/6, 10, 0, 1)$.

| Lie group | # vector fields | # motion primitives | systems | inversion |
|---|---|---|---|---|
| SE(2) | 2 | 3 | $\mathcal{S}_1$ | global |
| | | | $\mathcal{S}_2$ | local |
| SO(3) | 2 | 3 | all | local |
| $\mathrm{SE}(2)\times\mathbb{R}$ | 2 | 5 | $\mathcal{T}_1$ | global |
| | | | $\mathcal{T}_2$ | local |
| | 3 | 4 | $\mathcal{T}_3$ | global |
| | | | $\mathcal{T}_4$ | global |
| | | | $\mathcal{T}_5$ | local |

TABLE 1. Summary of results.

## REFERENCES

[1] H. Arai, K. Tanie, and N. Shiroma. Nonholonomic control of a three-DOF planar underactuated manipulator. *IEEE Transactions on Robotics and Automation*, 14(5):681–695, 1998.

[2] J. E. Bobrow, S. Dubowsky, and J. S. Gibson. Time-optimal control of robotic manipulators along specified paths. *International Journal of Robotics Research*, 4(3):3–17, 1985.

[3] F. Bullo and A. D. Lewis. Kinematic controllability and motion planning for the snakeboard. *IEEE Transactions on Robotics and Automation*, 19(3):494–498, 2003.

[4] F. Bullo and A. D. Lewis. *Geometric Control of Mechanical Systems*, volume 49 of *Texts in Applied Mathematics*. Springer, 2004.

[5] F. Bullo and A. D. Lewis. Low-order controllability and kinematic reductions for affine connection control systems. *SIAM Journal on Control and Optimization*, 44(3):885–908, 2005.

[6] F. Bullo and K. M. Lynch. Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems. *IEEE Transactions on Robotics and Automation*, 17(4):402–412, 2001.
[7] P. B. Davenport. Rotations about nonorthogonal axes. *AIAA Journal*, 11(6):853–857, 1973.
[8] S. S. Ge, S. Zhendong, T. H. Lee, and M. W. Spong. Feedback linearization and stabilization of second-order non-holonomic chained systems. *International Journal of Control*, 74(14):1383–92, 2001.
[9] A. Goldenberg, B. Benhabib, and R. Fenton. A complete generalized solution to the inverse kinematics of robots. *IEEE Journal of Robotics and Automation*, 1(1):14–20, 1985.
[10] D. Halperin, L. E. Kavraki, and J.-C. Latombe. Robotics. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 41, pages 755–778. CRC Press, 1997.
[11] S. Iannitti and K. M. Lynch. Minimum control-switch motions for the snakeboard: A case study in kinematically controllable underactuated systems. *IEEE Transactions on Robotics*, 20(6):994–1006, 2004.
[12] G. Lafferriere and H. J. Sussmann. A differential geometric approach to motion planning. In Z. Li and J. F. Canny, editors, *Nonholonomic Motion Planning*, pages 235–270. Kluwer Academic Publishers, 1993.
[13] N. E. Leonard and P. S. Krishnaprasad. Motion control of drift-free, left-invariant systems on Lie groups. *IEEE Transactions on Automatic Control*, 40(9):1539–1554, 1995.
[14] K. M. Lynch and M. T. Mason. Stable pushing: Mechanics, controllability, and planning. *International Journal of Robotics Research*, 15:533–556, 1996.
[15] K. M. Lynch, N. Shiroma, H. Arai, and K. Tanie. Collision-free trajectory planning for a 3-DOF robot with a passive joint. *International Journal of Robotics Research*, 19(12):1171–1184, 2000.
[16] D. Manocha and J. F. Canny. Efficient inverse kinematics for general 6R manipulators. *IEEE Transactions on Robotics and Automation*, 10(5):648–657, 1994.
[17] P. Martin and P. Rouchon. Feedback linearization and driftless systems. *Mathematics of Control, Signals and Systems*, 7(3):235–254, 1995.
[18] M. C. Muñoz-Lecanda and F. J. Yániz. Mechanical control systems and kinematic systems. *IEEE Transactions on Automatic Control*, 53(5):1297–1302, 2008.
[19] R. M. Murray, Z. X. Li, and S. S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.
[20] B. Paden. *Kinematics and Control of Robot Manipulators*. PhD thesis, UC Berkeley, 1986.
[21] C. Samson. Control of chained systems. Application to path following and time-varying point-stabilization of mobile robots. *IEEE Transactions on Automatic Control*, 40(1):64–77, 1995.
[22] L. Sciavicco and B. Siciliano. *Modeling and Control of Robot Manipulators*. Springer, 2 edition, 2000.
[23] O. J. Sordalen and O. Egeland. Exponential stabilization of nonholonomic chained systems. *IEEE Transactions on Automatic Control*, 40(1):35–49, 1995.
[24] M. W. Spong and M. Vidyasagar. *Robot Dynamics and Control*. John Wiley, 1989.
[25] J. Wittenburg and L. Lilov. Decomposition of a finite rotation into three rotations about given axes. *Multibody System Dynamics*, 9(4):353–375, 2003.

Received xxxx 20xx; revised xxxx 20xx.

*E-mail address*: soniamd@ucsd.edu
*E-mail address*: cortes@ucsd.edu
*E-mail address*: bullo@engineering.ucsb.edu