On the convergence time of asynchronous distributed quantized averaging algorithms

Minghui Zhu and Sonia Martínez

Abstract

We come up with a class of distributed quantized averaging algorithms on asynchronous communication networks with fixed and switching topologies. The focus of this paper is on the study of the convergence time of the proposed quantized averaging algorithms. By appealing to random walks on graphs, we derive polynomial bounds on the expected convergence time of the algorithms presented, as a function of the number of agents in the network.

I. INTRODUCTION

Consider a network of N (mobile or immobile) agents. The distributed consensus problem aims to design an algorithm that agents can utilize to asymptotically reach an agreement by communicating with nearest neighbors. This problem historically roots in parallel computation [2], and has attracted significant attention recently [3][11][16]. As a special case of the consensus problem, the distributed averaging problem requires that the consensus value be the average of individual initial states.

In real-world communication networks, the capacities of communication channels and the memory capacities of agents are finite. Furthermore, the computations can only be carried out with finite precision. From a practical point of view, real-valued averaging algorithms are not feasible and these realistic constraints motivate the problem of average consensus via quantized information. Another motivation for distributed quantized averaging is load balancing with indivisible tasks. Prior work on distributed quantized averaging over fixed graphs includes [1], [6], [7], [12]. Recently, [15] examines quantization effects on distributed averaging algorithms over

The authors are with Department of Mechanical and Aerospace Engineering, University of California, San Diego, 9500 Gilman Dr, La Jolla CA, 92093, {`mizhu,soniamd`}`@ucsd.edu`

time-varying topologies. As in [12], we focus on gossip-based quantized averaging algorithms preserving the sum of the state values at each iteration. This setup has the following properties of interest: the sum cannot be changed in some situations, such as load balancing; and the constant sum leads to a small steady-state error with respect to the average of individual initial states. This error is equal to either one quantization step size or zero (when the average of the initial states is located at one of the quantization levels) and thus is independent of N.

The worst-case upper bound, as a function of N and independent of network topologies, on the convergence time is a typical measure to quantify the performance of real-valued averaging (e.g., in [17]) and quantized averaging (e.g., in [12], [15]). In particular, polynomial bounds over fixed complete and linear graphs are derived in [12]; and, the authors in [15] give a polynomial bound over switching topologies. To explicitly find these bounds, the paper [15] requires a common time-slotted system, and [12] needs some global information (e.g, a centralized entity choosing one edge to establish communication or the number of the edges available in the graph). However, real-world communication networks are inherently asynchronous [1] environment and lack of centralized coordination. These constraints motivate us to employ the asynchronous network model in [4] to select one edge at each time instant in a totally distributed manner without any global information.

After the submission of this work, the papers [9], [13] study gossip-based quantized averaging algorithms and characterize the bounds on their convergence rates for fixed and connected graphs. Instead of being functions of N, these bounds depend on graph topologies. Our results, developed independently from [9], [13], provide general bounds on the expected convergence time over fixed and switching topologies.

Statement of contributions. The present paper proposes a class of distributed quantized averaging algorithms on asynchronous communication networks with fixed and switching topologies. The algorithms are shown to asymptotically reach quantized average consensus almost surely. Furthermore, we utilize meeting times of two random walks on graphs as a unified approach to derive polynomial bounds on their expected convergence times. Toward our best knowledge, this note is the first time to derive such polynomial bounds without requiring any global information. We refer the readers to [14] for an enlarged version of this note that includes all the proofs. A

---

[1] As in [4], asynchronism in this note means that time is not assumed to be slotted commonly across nodes.

preliminary conference version of this paper is in [19] where the convergence time of synchronous algorithms is also studied.

II. PRELIMINARIES AND PROBLEM STATEMENT

Here, we present the problem formulation along with some notation and terminology.

Asynchronous time model. In this note, we will employ the asynchronous time model proposed in [4]. More precisely, consider a network of N nodes, labeled 1 through N. Each node has a clock which ticks according to a rate 1 Poisson process. Hence, the inter-tick times at each node are random variables with rate 1 exponential distribution, independent across nodes and independent over time. By the superposition theorem for Poisson processes, this setup is equivalent to a single global clock modeled as a rate N Poisson process ticking at times $\{Z_k\}_{k\geq 0}$. By the orderliness property of Poisson processes, the clock ticks do not occur simultaneously. The inter-agent communication and the update of consensus states only occur at $\{Z_k\}_{k\geq 0}$. In the reminder of this paper, the time instant t will be discretized according to $\{Z_k\}_{k\geq 0}$ and defined in terms of the number of clock ticks.

Network model. We will employ the undirected graph $\mathcal{G}(t) = (V, E(t))$ to model the network. Here $V := \{1, \cdots, N\}$ is the vertex set, and an edge $(j, i) \in E(t)$ if and only if node j can receive the message from node i (e.g., node j is within the communication range of node i) at time t. The neighbors of node i at time t are denoted by $\mathcal{N}_i(t) = \{j \in V \mid (j, i) \in E(t) \text{ and } j \neq i\}$. The state of node i at time t is denoted by $x_i(t) \in \mathbb{R}$ and the network state is denoted by $x(t) = (x_1(t), \cdots, x_N(t))^T$. Suppose the initial states $x_i(0) \in [U_{\min}, U_{\max}]$ for all $i \in V$ and some real numbers $U_{\min}$ and $U_{\max}$.

Quantization scheme. Let R denote the number of bits per sample. The total number of quantization levels can be represented by $L = 2^R$ and the step size is $\Delta = (U_{\max} - U_{\min})/2^R$. The quantization levels, $\{\omega_1, \cdots, \omega_L\}$, are uniformly spaced in the sense that $\omega_{i+1} - \omega_i = \Delta$ for $i \in \{1, \cdots, L-1\}$. A quantizer $\mathcal{Q} : [U_{\min}, U_{\max}] \to \{\omega_1, \cdots, \omega_L\}$ is adopted to quantize the message $u \in [U_{\min}, U_{\max}]$ in such a way that $\mathcal{Q}(u) = \omega_i$ if $u \in [\omega_i, \omega_{i+1})$ for some $i \in \{1, \cdots, L-1\}$. Assume that the initial states $x_i(0)$ for all $i \in V$ are multiples of Δ.

Problem statement. The problem of interest in this paper is to design distributed averaging algorithms which the nodes can utilize to update their states by communicating with neighbors via quantized messages in an asynchronous setting. Ultimately, quantized average consensus is reached in probability; i.e., for any initial state $x(0)$, there holds that $\lim_{t\to\infty}\mathbb{P}(x(t) \in \mathcal{W}(x(0))) = 1$. The set $\mathcal{W}(x(0))$ is dependent on initial state $x(0) \in \mathbb{R}^N$ and defined as follows. If $\bar{x}(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$ is not a multiple of Δ, then $\mathcal{W}(x(0)) = \{x \in \mathbb{R}^N \mid x_i \in \{\mathcal{Q}(\bar{x}(0)), \mathcal{Q}(\bar{x}(0)) + \Delta\}\}$; otherwise, $\mathcal{W}(x(0)) = \{x \in \mathbb{R}^N \mid x_i = \bar{x}(0)\}$. Now it is clear that the steady-state error is at most Δ after quantized average consensus is reached.

Notions of random walks on graphs. In this paper, random walks on graphs play an important role in characterizing the convergence properties of our quantized averaging algorithms. The following definitions are generalized from those defined for fixed graphs in [5], [8].

Definition 2.1 (Random walks): A random walk on the graph $\mathcal{G}(t)$ under the transition matrix $P(t) = (p_{ij}(t))$, starting from node v at time s, is a stochastic process $\{X(t)\}_{t\geq s}$ such that $X(s) = v$ and $\mathbb{P}(X(t+1) = j \mid X(t) = i) = p_{ij}(t)$. A random walk is said to be simple if for any $i \in V$, $p_{ii}(t) = 0$ for all $t \geq 0$; otherwise, it is said to be natural. •

Definition 2.2 (Hitting time): Consider a random walk on the graph $\mathcal{G}(t)$, beginning from node i at time s and evolving under the transition matrix $P(t)$. The hitting time from node i to the set $\Lambda \subseteq V$, denoted as $H_{(\mathcal{G}(t),P(t),s)}(i,\Lambda)$, is the expected time it takes this random walk to reach the set Λ for the first time. We denote $H_{(\mathcal{G}(t),P(t))}(\Lambda) = \sup_{s\geq 0}\max_{i\in V} H_{(\mathcal{G}(t),P(t),s)}(i,\Lambda)$ as the hitting time to reach the set Λ. The hitting time of the pair i, j, denoted as $H_{(\mathcal{G}(t),P(t),s)}(i,j)$, is the expected time it takes this random walk to reach node j for the first time. Denote $H_{(\mathcal{G}(t),P(t))} = \sup_{s\geq 0}\max_{i,j\in V} H_{(\mathcal{G}(t),P(t),s)}(i,j)$ as the hitting time of going between any pair of nodes. •

Definition 2.3 (Meeting time): Consider two random walks on the graph $\mathcal{G}(t)$ under the transition matrix $P(t)$, starting at time s from node i and node j respectively. The meeting time $M_{(\mathcal{G}(t),P(t),s)}(i,j)$ of these two random walks is the expected time it takes them to meet at some node for the first time. The meeting time on the graph $\mathcal{G}(t)$ is defined as $M_{(\mathcal{G}(t),P(t))} = \sup_{s\geq 0}\max_{i,j\in V} M_{(\mathcal{G}(t),P(t),s)}(i,j)$. •

For the ease of notation, we will drop the subscript s in the hitting time and meeting time notions for fixed graphs.

Notations. For $\alpha \in \mathbb{R}$, define $V_\alpha : \mathbb{R}^N \to \mathbb{R}$ as $V_\alpha(x) = \sum_{i=1}^{N}(x_i - \alpha)^2$. We define $J : \mathbb{R}^N \to \mathbb{R}$ as $J(x) = (\max_{i\in V} x_i - \min_{i\in V} x_i)/\Delta$. Denote the set $\Theta = \{(k,k) \mid k \in V\}$. The $distribution$ of a vector $x \in \mathbb{R}^N$ is defined to be the list $\{(q_1, m_1), (q_2, m_2), \cdots, (q_k, m_k)\}$ for some $k \in V$ where $\sum_{\ell=1}^{k} m_\ell = N$, $q_i \neq q_j$ for $i \neq j$ and m_ℓ is the cardinality of the set $\{i \in V \mid x_i = q_\ell\}$. The cardinality of the set M is denoted by $|M|$.

III. ASYNCHRONOUS DISTRIBUTED QUANTIZED AVERAGING ON FIXED GRAPHS

In this section, we propose and analyze an asynchronous distributed quantized averaging algorithm on the fixed and connected graph $\mathcal{G}$. Main references are [12] on quantized gossip algorithms and [5] on the meeting time of two simple random walks on fixed graphs.

A. Proposed algorithm

The *asynchronous distributed quantized averaging algorithm on the fixed and connected graph* $\mathcal{G}$ (AF, for short) is described as follows. Suppose node i's clock ticks at time t. Node i randomly chooses one of its neighbors, say node j, with equal probability. Node i and j then execute the following local computation. If $x_i(t) \geq x_j(t)$, then

$$x_i(t+1) = x_i(t) - \delta, \quad x_j(t+1) = x_j(t) + \delta; \tag{1}$$

otherwise,

$$x_i(t+1) = x_i(t) + \delta, \quad x_j(t+1) = x_j(t) - \delta, \tag{2}$$

where $\delta = \frac{1}{2}|x_i(t) - x_j(t)|$ if $\frac{|x_i(t)-x_j(t)|}{2\Delta}$ is an integer; otherwise, $\delta = \mathcal{Q}(\frac{1}{2}|x_i(t) - x_j(t)|) + \Delta$. Every other node $k \in V \setminus \{i, j\}$ preserves its current state; i.e., $x_k(t+1) = x_k(t)$.

Remark 3.1: The precision $\frac{\Delta}{2}$ is sufficient for the computation of δ and thus the update laws (1) and (2). It is easy to verify that $x_i(t) \in [U_{\min}, U_{\max}]$ and $x_i(t)$ are multiples of Δ for all $i \in V$ and $t \geq 0$. Furthermore, the sum of the state values is preserved at each iteration.

If $|x_i(t) - x_j(t)| = \Delta$, the update laws (1) and (2) become $x_i(t+1) = x_j(t)$ and $x_j(t+1) = x_i(t)$. Such update is referred to as a *trivial average* in [12]. If $|x_i(t) - x_j(t)| > \Delta$, then (1) or (2) is referred to as a *non-trivial average*. Although it does not directly contribute to reaching quantized average consensus, trivial average helps the information flow over the network. •

B. The meeting time of two natural random walks on the fixed graph $\mathcal{G}$

We first study a variation of the problem in [8], namely, *the meeting time of two natural random walks on the fixed graph* $\mathcal{G}$. More precisely, assume that the fixed graph $\mathcal{G}$ be undirected and connected. Initially, two tokens are placed on the graph $\mathcal{G}$; at each time, one of the tokens is chosen with probability $\frac{1}{N}$ and the chosen token moves to one of the neighboring nodes with equal probability. What is the meeting time for these two tokens?

The tokens move as two natural random walks with the transition matrix P_{AF} on the graph $\mathcal{G}$. The matrix $P_{\mathrm{AF}} = (\tilde{p}_{ij}) \in \mathbb{R}^{N\times N}$ is given by $\tilde{p}_{ii} = 1 - \frac{1}{N}$ for $i \in V$, $\tilde{p}_{ij} = \frac{1}{N|\mathcal{N}_i|}$ for $(i,j) \in E$. Their meeting time is denoted as $M_{(\mathcal{G},P_{\mathrm{AF}})}$. Denote any of these two natural random walks as $X_{\mathcal{N}}$. Correspondingly, we construct a simple random walk, say $X_{\mathcal{S}}$, with the transition matrix P_{SF} on the graph $\mathcal{G}$ where $P_{\mathrm{SF}} = (p_{ij}) \in \mathbb{R}^{N\times N}$ is given by $p_{ii} = 0$ and $p_{ij} = \frac{1}{|\mathcal{N}_i|}$ if $(i,j) \in E$. The hitting times of $X_{\mathcal{S}}$ and $X_{\mathcal{N}}$ are denoted as $H_{(\mathcal{G},P_{\mathrm{SF}})}$ and $H_{(\mathcal{G},P_{\mathrm{AF}})}$, respectively.

Proposition 3.1: For the problem of the meeting time of two natural random walks on the fixed graph $\mathcal{G}$, it holds that $M_{(\mathcal{G},P_{\mathrm{AF}})} \leq 2NH_{(\mathcal{G},P_{\mathrm{SF}})} - N$.

Proof: Since the fixed graph $\mathcal{G}$ is undirected and connected, the random walks $X_{\mathcal{N}}$ and $X_{\mathcal{S}}$ are irreducible. The reminder of the proof is based on the following claims:

(i) It holds that $H_{(\mathcal{G},P_{\mathrm{AF}})} \geq N$.

(ii) For any pair $i, j \in V$ with $i \neq j$, we have $H_{(\mathcal{G},P_{\mathrm{AF}})}(i,j) = NH_{(\mathcal{G},P_{\mathrm{SF}})}(i,j)$.

(iii) For any $i, j, k \in V$, the following equality holds:

$H_{(\mathcal{G},P_{\mathrm{AF}})}(i,j) + H_{(\mathcal{G},P_{\mathrm{AF}})}(j,k) + H_{(\mathcal{G},P_{\mathrm{AF}})}(k,i) = H_{(\mathcal{G},P_{\mathrm{AF}})}(i,k) + H_{(\mathcal{G},P_{\mathrm{AF}})}(k,j) + H_{(\mathcal{G},P_{\mathrm{AF}})}(j,i)$.

(iv) There holds that $M_{(\mathcal{G},P_{\mathrm{AF}})} \leq 2H_{(\mathcal{G},P_{\mathrm{AF}})} - N$.

Due to the space limitation, we omit the details of the proofs. ■

C. *Convergence analysis of* AF

We now proceed to analyze the convergence properties of AF. The convergence time of AF is a random variable defined as follows: $T_{\mathrm{con}}(x(0)) = \inf\{t \mid x(t) \in \mathcal{W}(x(0))\}$, where $x(t)$ starts from $x(0)$ and evolves under AF. Choose $V_{\bar{x}(0)}(x) = \sum_{i=1}^{N}(x_i - \bar{x}(0))^2$ as a Lyapunov function candidate for AF. One can readily see that $V_{\bar{x}(0)}(x(t+1)) = V_{\bar{x}(0)}(x(t))$ when a trivial average occurs and $V_{\bar{x}(0)}(x)$ reduces at least $2\Delta^2$ when a non-trivial average occurs. Hence, $V_{\bar{x}(0)}(x)$ is non-increasing along the trajectories, and the number of non-trivial averages is at most $\frac{1}{2\Delta^2}V_{\bar{x}(0)}(x(0))$. Define the set $\Psi = \{x \in \mathbb{R}^N \mid \text{the distribution of } x \text{ is } \{(0,1), (\Delta, N-2), (2\Delta, 1)\}\}$ and denote $\mathbb{E}[T_\Psi] = \max_{x(0)\in\Psi} \mathbb{E}[T_{\mathrm{con}}(x(0))]$. It is clear that the expected time between any two consecutive non-trivial averages is not larger than $\mathbb{E}[T_\Psi]$. Then we have the following estimates on $\mathbb{E}[T_{\mathrm{con}}(x(0))]$:

$$\mathbb{E}[T_{\mathrm{con}}(x(0))] \leq \frac{1}{2\Delta^2}V_{\bar{x}(0)}(x(0))\mathbb{E}[T_\Psi] \leq \frac{NJ(x(0))^2}{8}\mathbb{E}[T_\Psi], \tag{3}$$

where the second inequality is a direct result of Lemma 4 in [12].

Theorem 3.1: For any $x(0) \notin \mathcal{W}(x(0))$, the expected convergence time $\mathbb{E}[T_{\text{con}}(x(0))]$ of AF is upper bounded by $\frac{N^2 J(x(0))^2}{8}(\frac{8}{27}N^3 - 1)$. Furthermore, almost any evolution $x(t)$ starting from $x(0)$ reaches quantized average consensus.

Proof: By (3), it suffices to bound $\mathbb{E}[T_\Psi]$. Assume that $x(0) \in \Psi$. Before they meet for the first time, the values 0 and 2Δ move as two natural random walks which are identical to $X_{\mathcal{N}}$ in Proposition 3.1. At their meeting for the first time, the values of 0 and 2Δ average and quantized average consensus is reached. Hence, $\mathbb{E}[T_\Psi] = M_{(\mathcal{G}, P_{\text{AF}})}$ and thus inequality (3) becomes

$$\mathbb{E}[T_{\text{con}}(x(0))] \leq \frac{NJ(x(0))^2}{8} M_{(\mathcal{G}, P_{\text{AF}})} \leq \frac{NJ(x(0))^2}{8}(2NH_{(\mathcal{G}, P_{\text{SF}})} - N), \tag{4}$$

where we use Proposition 3.1 in the second inequality. Substituting the upper bound on $H_{(\mathcal{G}, P_{\text{SF}})}$ in [5] into inequality (4) gives the desired upper bound on $\mathbb{E}[T_{\text{con}}(x(0))]$. The remainder of the proofs can be completed by using the property of the set $\mathcal{W}(x(0))$ being absorbing and Markov's inequality. We omit the details here. ■

IV. ASYNCHRONOUS DISTRIBUTED QUANTIZED AVERAGING ON SWITCHING GRAPHS

We now turn our attention to the more challenging scenario where the communication graphs are undirected but dynamically changing. We will propose and analyze an *asynchronous distributed quantized averaging algorithm on switching graphs* (AS, for short). The convergence rate of distributed real-valued averaging algorithms on switching graphs in [15] will be employed to characterize the hitting time of random walks on switching graphs.

A. Proposed algorithm

The main steps of AS can be summarized as follows. At time t, let node i's clock tick. If $|\mathcal{N}_i(t)| \neq 0$, node i randomly chooses one of its neighbors, say node j, with probability $\frac{1}{\max\{|\mathcal{N}_i(t)|, |\mathcal{N}_j(t)|\}}$. Then, node i and j execute the computation (1) or (2) and every other node $k \in V \backslash \{i, j\}$ preserves its current state. If $|\mathcal{N}_i(t)| = 0$, all the nodes do nothing at this time.

Here, we assume that the communication graph $\mathcal{G}(t)$ be undirected and satisfies the following connectivity assumption also used in [3], [11], [15], [17].

Assumption 4.1 (Periodical connectivity): There exists some $B \in \mathbb{N}_{>0}$ such that, for all $t \geq 0$, the undirected graph $(V, E(t) \cup E(t+1) \cup \cdots \cup E(t+B-1))$ is connected.

B. The meeting time of two natural random walks on the time-varying graph $\mathcal{G}(t)$

Before analyzing AS, we consider the following problem which generalizes the problem in Section III-B to the case of dynamically changing graphs.

The meeting time of two natural random walks on the time-varying graph $\mathcal{G}(t)$. Assume that $\mathcal{G}(t)$ be undirected and satisfies Assumption 4.1. Initially, two tokens are placed on $\mathcal{G}(0)$. At each time, one of the tokens is chosen with probability $\frac{1}{N}$. The chosen token at some node, say i, moves to one of the neighbors, say node j, with probability $\frac{1}{\max\{|\mathcal{N}_i(t)|,|\mathcal{N}_j(t)|\}}$ if $|\mathcal{N}_i(t)| \neq 0$; otherwise, it will stay up with node i. What is the meeting time for these two tokens?

Clearly, the movements of two tokens are two natural random walks, say X_1 and X_2, on the switching graph $\mathcal{G}(t)$. Their meeting time is denoted as $M_{(\mathcal{G}(t),P_{\mathrm{AS}}(t))}$ where the transition matrix $P_{\mathrm{AS}}(t) = (\bar{p}_{ij}(t))$ is given as follows: if $|\mathcal{N}_i(t)| \neq 0$, then $\bar{p}_{ij}(t) = \frac{1}{N\max\{|\mathcal{N}_i(t)|,|\mathcal{N}_j(t)|\}}$ for $(i,j) \in E(t)$ and $\bar{p}_{ii}(t) = 1 - \sum_{(i,j)\in E(t)} \frac{1}{N\max\{|\mathcal{N}_i(t)|,|\mathcal{N}_j(t)|\}}$; if $|\mathcal{N}_i(t)| = 0$, then $\bar{p}_{ii}(t) = 1$. One can easily verify that the matrix $P_{\mathrm{AS}}(t)$ is symmetric and doubly stochastic. The natural random walks X_1 and X_2 on the graph $\mathcal{G}(t)$ are equivalent to a single natural random walk, say X_M, on the product graph $\mathcal{G}(t) \times \mathcal{G}(t)$. That is, X_M moving from node $(i_1,i_2) \in V \times V$ to node $(j_1,j_2) \in V \times V$ on the graph $\mathcal{G}(t) \times \mathcal{G}(t)$ at time t, is equivalent to X_1 moving from i_1 to j_1 and X_2 moving from i_2 to j_2 on the graph $\mathcal{G}(t)$ at time t. Denote the transition matrix of the random walk X_M as $Q(t) = (q_{(i_1,i_2)(j_1,j_2)}(t)) \in \mathbb{R}^{N^2\times N^2}$.

In the following lemma, we will consider the random walk $\bar{X}_M$ on the graph $\mathcal{G}(t) \times \mathcal{G}(t)$ with the absorbing set Θ and the transition matrix $\bar{Q}(t) \in \mathbb{R}^{N^2\times N^2}$. Denote $e_{(\ell_1,\ell_2)}$ by the row corresponding to $(\ell_1,\ell_2) \in V \times V$ in a $N^2 \times N^2$ identity matrix. The transition matrix $\bar{Q}(t)$ is defined by replacing the row associated with the absorbing state $(\ell_1,\ell_2) \in \Theta$ in $Q(t)$ with $e_{(\ell_1,\ell_2)}$. Define $\vartheta_{(\ell_1,\ell_2)}(t) = \mathbb{P}(X_M(t) = (\ell_1,\ell_2))$, $\vartheta(t) = \mathrm{col}\{\vartheta_{(\ell_1,\ell_2)}(t)\} \in \mathbb{R}^{N^2}$, $\vartheta_\Theta(t) = \sum_{(\ell_1,\ell_2)\in\Theta} \vartheta_{(\ell_1,\ell_2)}(t)$ for the random walk X_M, and $\bar{\vartheta}_{(\ell_1,\ell_2)}(t) = \mathbb{P}(\bar{X}_M(t) = (\ell_1,\ell_2))$, $\bar{\vartheta}(t) = \mathrm{col}\{\bar{\vartheta}_{(\ell_1,\ell_2)}(t)\} \in \mathbb{R}^{N^2}$, $\bar{\vartheta}_\Theta(t) = \sum_{(\ell_1,\ell_2)\in\Theta} \bar{\vartheta}_{(\ell_1,\ell_2)}(t)$ for the random walk $\bar{X}_M$.

Lemma 4.1: Consider a network of N nodes whose communication graph $\mathcal{G}(t)$ be undirected and satisfies Assumption 4.1. Let $(i_1,i_2) \in V \times V$ be a given node and suppose that the random walks X_M and $\bar{X}_M$ start from node (i_1,i_2) at time 0. Then it holds that $\bar{\vartheta}_\Theta(t) \geq \vartheta_\Theta(t) \geq \frac{1}{2N}$ for $t \geq t_1$ where $t_1 := B(8N^6\log(\sqrt{2}N) + 1)$.

Proof: It is not difficult to check that $\mathcal{G}(t)\times\mathcal{G}(t)$ is undirected and satisfies Assumption 4.1

with period B. The minimum of nonzero entries in $Q(t)$ is lower bounded by $\frac{1}{N(N-1)}$, and $Q(t)$ is symmetric. Observe that for any $(i_1, i_2) \in V \times V$ and any $t \geq 0$, $\sum_{(j_1,j_2)\in V\times V} q_{(i_1,i_2)(j_1,j_2)}(t) = \sum_{(j_1,j_2)\in V\times V} \bar{p}_{i_1 j_1}(t)\bar{p}_{i_2 j_2}(t) = \sum_{j_1\in V} \bar{p}_{i_1 j_1}(t) \times \sum_{j_2 \in V} \bar{p}_{i_2 j_2}(t) = 1$ where we use the fact that the matrix $P_{\text{AS}}(t)$ is doubly stochastic. Hence, the matrix $Q(t)$ is doubly stochastic.

The evolution of $\vartheta(t)$ is governed by the equation $\vartheta(t+1) = Q^T(t)\vartheta(t)$ with initial state $\vartheta(0) = e^T_{(i_1,i_2)}$. Consider the Lyapunov function $V_{\frac{1}{N^2}}(\vartheta) = \sum_{i=1}^{N^2}(\vartheta_i - \frac{1}{N^2})^2$ with $V_{\frac{1}{N^2}}(\vartheta(0)) = 1 - \frac{1}{N^2}$. It follows from Lemma 5 in [15] that

$$V_{\frac{1}{N^2}}(\vartheta((k+1)B)) \leq (1 - \frac{1}{2N^5(N-1)})V_{\frac{1}{N^2}}(\vartheta(kB)) \tag{5}$$

for $k \in \mathbb{N}_0$. Denote $\mathbf{1} \in \mathbb{R}^{N^2}$ as the vector of N^2 ones and note that

$$V_{\frac{1}{N^2}}(\vartheta(t)) - V_{\frac{1}{N^2}}(\vartheta(t+1)) = (\vartheta(t) - \frac{1}{N^2}\mathbf{1})^T(I - Q(t)Q^T(t))(\vartheta(t) - \frac{1}{N^2}\mathbf{1}).$$

Since $Q(t)$ is doubly stochastic, so is $Q(t)Q^T(t)$. Hence, the diagonal entries of the matrix $\Gamma(t) = I - Q(t)Q^T(t) = (\gamma_{ij}(t)) \in \mathbb{R}^{N^2\times N^2}$ are dominant in the sense of $\gamma_{ii}(t) = \sum_{j\neq i}\gamma_{ij}(t)$. According to Gershgorin theorem in [10], all eigenvalues of $\Gamma(t)$ lie in a closed disk centered at $\max_{i\in\{1,\cdots,N^2\}}\gamma_{ii}(t)$ with a radius $\max_{i\in\{1,\cdots,N^2\}}\gamma_{ii}(t)$. Hence, $\Gamma(t)$ is positive semi-definite, and thus $V_{\frac{1}{N^2}}(\vartheta(t))$ is non-increasing along the trajectory of $\vartheta(t)$. Combining (5) with the non-increasing property of $V_{\frac{1}{N^2}}(\vartheta(t))$ gives that

$$V_{\frac{1}{N^2}}(\vartheta(t)) \leq V_{\frac{1}{N^2}}(\vartheta(0))(1 - \frac{1}{2N^5(N-1)})^{\frac{t}{B}-1} = \frac{N^2-1}{N^2}(1 - \frac{1}{2N^5(N-1)})^{\frac{t}{B}-1}. \tag{6}$$

Since $\vartheta(t)^T\mathbf{1} = 1$, then $\vartheta_{\min}(t) := \min_{(\ell_1,\ell_2)\in V\times V}\vartheta_{(\ell_1,\ell_2)}(t) \leq \frac{1}{N^2}$. Since $V_{\frac{1}{N^2}}(\vartheta(t)) \geq (\vartheta_{\min}(t) - \frac{1}{N^2})^2$, inequality (6) gives that $\vartheta_{\min}(t) \geq \frac{1}{N^2} - (\frac{N^2-1}{N^2}(1 - \frac{1}{2N^5(N-1)})^{\frac{t}{B}-1})^{\frac{1}{2}}$. Therefore, it holds that $\vartheta_{\min}(t) \geq \frac{1}{2N^2}$ for $t \geq B(\frac{\log(4N^2(N^2-1))}{-\log(1-\frac{1}{2N^5(N-1)})} + 1)$. Since $\log x \leq x - 1$, there holds $\frac{1}{-\log(1-\frac{1}{2N^5(N-1)})} \leq 2N^5(N-1) \leq 2N^6$. Hence, we have that $\vartheta_{\min}(t) \geq \frac{1}{2N^2}$ and thus $\vartheta_{\Theta}(t) \geq \frac{1}{2N}$ for $t \geq t_1$.

Note that the evolution of $\bar{\vartheta}(t)$ is governed by $\bar{\vartheta}(t+1) = \bar{Q}(t)^T\bar{\vartheta}(t)$ with $\bar{\vartheta}(0) = e_{(i_1,i_2)}$. Since the set Θ is absorbing, $\bar{\vartheta}_{\Theta}(t) \geq \vartheta_{\Theta}(t)$ for all $t \geq 0$ and thus the desired result follows. ■

Proposition 4.1: For the problem of the meeting time of two natural random walks on the time-varying graph $\mathcal{G}(t)$, it holds that $M_{(\mathcal{G}(t),P_{\text{AS}}(t))} \leq 4Nt_1$.

Proof: Denote by $H_{(\mathcal{G}(t)\times\mathcal{G}(t),Q(t))}(\Theta)$ the hitting time of the random walk X_M to reach the set of Θ. Observe that $M_{(\mathcal{G}(t),P_{\text{AS}}(t))} = H_{(\mathcal{G}(t)\times\mathcal{G}(t),Q(t))}(\Theta)$. To find an upper bound on

$H_{(\mathcal{G}(t)\times\mathcal{G}(t),Q(t))}(\Theta)$, we construct the random walk $X_M^{(i_1,i_2)}$ in such a way that $X_M^{(i_1,i_2)}$ starts from (i_1,i_2) at time 0 with $i_1\neq i_2$ and the set Θ is the absorbing set of $X_M^{(i_1,i_2)}$. The transition matrix of $X_M^{(i_1,i_2)}$ is $\bar{Q}(t)$ defined before Lemma 4.1. Define $\vartheta^{(i_1,i_2)}_{(\ell_1,\ell_2)}(t)=\mathbb{P}(X_M^{(i_1,i_2)}(t)=(\ell_1,\ell_2))$, and $\vartheta^{(i_1,i_2)}(t)=\mathrm{col}\{\vartheta^{(i_1,i_2)}_{(\ell_1,\ell_2)}(t)\}\in\mathbb{R}^{N^2}$. The dynamics of $\vartheta^{(i_1,i_2)}(t)$ is given by $\vartheta^{(i_1,i_2)}(t+1)=\bar{Q}(t)^T\vartheta^{(i_1,i_2)}(t)$ with the initial state $\vartheta^{(i_1,i_2)}(0)=e^T_{(i_1,i_2)}$.

Define the function $\mu^{(i_1,i_2)}_{(\ell_1,\ell_2)}:\mathbb{N}_0\to\{0,1\}$ in such a way that $\mu^{(i_1,i_2)}_{(\ell_1,\ell_2)}=1$ if $X_M^{(i_1,i_2)}(t)=(\ell_1,\ell_2)$; otherwise, $\mu^{(i_1,i_2)}_{(\ell_1,\ell_2)}(t)=0$. Then, the hitting time $H_{(\mathcal{G}(t)\times\mathcal{G}(t),Q(t),0)}((i_1,i_2),\Theta)$ of $X_M^{(i_1,i_2)}$ equals the expected time that $X_M^{(i_1,i_2)}$ stays up with the nodes in $V\times V\backslash\Theta$, that is,

$$H_{(\mathcal{G}(t)\times\mathcal{G}(t),Q(t),0)}((i_1,i_2),\Theta)=\sum_{(\ell_1,\ell_2)\notin\Theta}\mathbb{E}[\sum_{\tau=0}^{+\infty}\mu^{(i_1,i_2)}_{(\ell_1,\ell_2)}(\tau)]=\sum_{\tau=0}^{+\infty}\sum_{(\ell_1,\ell_2)\notin\Theta}\vartheta^{(i_1,i_2)}_{(\ell_1,\ell_2)}(\tau). \tag{7}$$

It follows from Lemma 4.1 that $\vartheta^{(i_1,i_2)}_{\Theta}(t)\geq\frac{1}{2N}$ for $t\geq t_1$. With that, the fact of $\vartheta^{(i_1,i_2)}(t)^T\mathbf{1}=1$ implies that

$$\sum_{(\ell_1,\ell_2)\notin\Theta}\vartheta^{(i_1,i_2)}_{(\ell_1,\ell_2)}(t_1)\leq 1-\frac{1}{2N}. \tag{8}$$

For each $(k_1,k_2)\notin\Theta$, we construct the random walk $\tilde{X}_M^{(k_1,k_2)}$ in such a way that $\tilde{X}_M^{(k_1,k_2)}$ starts from (k_1,k_2) at time t_1 and the set Θ is the absorbing set of $\tilde{X}_M^{(k_1,k_2)}$. The transition matrix of $\tilde{X}_M^{(k_1,k_2)}$ is $\bar{Q}(t)$. Define $\tilde{\vartheta}^{(k_1,k_2)}_{(\ell_1,\ell_2)}(t)=\mathbb{P}(\tilde{X}_M^{(k_1,k_2)}(t)=(\ell_1,\ell_2))$. Following the forgoing arguments for $X_M^{(i_1,i_2)}$, we have

$$\sum_{(\ell_1,\ell_2)\notin\Theta}\tilde{\vartheta}^{(k_1,k_2)}_{(\ell_1,\ell_2)}(2t_1)\leq 1-\frac{1}{2N}. \tag{9}$$

Combining (8) and (9) gives that

$$\begin{aligned}\sum_{(\ell_1,\ell_2)\notin\Theta}\vartheta^{(i_1,i_2)}_{(\ell_1,\ell_2)}(2t_1)&=\sum_{(\ell_1,\ell_2)\notin\Theta}\sum_{(k_1,k_2)\notin\Theta}\vartheta^{(i_1,i_2)}_{(k_1,k_2)}(t_1)\tilde{\vartheta}^{(k_1,k_2)}_{(\ell_1,\ell_2)}(2t_1)\\&=\sum_{(k_1,k_2)\notin\Theta}\vartheta^{(i_1,i_2)}_{(k_1,k_2)}(t_1)\sum_{(\ell_1,\ell_2)\notin\Theta}\tilde{\vartheta}^{(k_1,k_2)}_{(\ell_1,\ell_2)}(2t_1)\leq(1-\frac{1}{2N})^2.\end{aligned} \tag{10}$$

By induction, we have $\sum_{(\ell_1,\ell_2)\notin\Theta}\vartheta^{(i_1,i_2)}_{(\ell_1,\ell_2)}(nt_1)\leq(1-\frac{1}{2N})^n$ and then obtain a strictly decreasing sequence $\sum_{(\ell_1,\ell_2)\notin\Theta}\vartheta^{(i_1,i_2)}_{(\ell_1,\ell_2)}(nt_1)$ with respect to $n\in\mathbb{Z}_0$. Since the set Θ is absorbing, then $\sum_{(\ell_1,\ell_2)\notin\Theta}\vartheta^{(i_1,i_2)}_{(\ell_1,\ell_2)}(t)$ is non-increasing with respect to $t\geq 0$. Therefore, we have the following estimate

$$\sum_{(\ell_1,\ell_2)\notin\Theta}\vartheta^{(i_1,i_2)}_{(\ell_1,\ell_2)}(t)\leq\sum_{(\ell_1,\ell_2)\notin\Theta}\vartheta^{(i_1,i_2)}_{(\ell_1,\ell_2)}(0)(1-\frac{1}{2N})^{\frac{t}{t_1}-1}=(1-\frac{1}{2N})^{\frac{t}{t_1}-1}. \tag{11}$$

Substituting (11) into (7) gives that

$$H_{(\mathcal{G}(t)\times\mathcal{G}(t),Q(t),0)}((i_1,i_2),\Theta) \leq \sum_{\tau=0}^{+\infty}(1-\frac{1}{2N})^{\frac{\tau}{t_1}-1} = (1-\frac{1}{2N})^{-\frac{1}{t_1}} \cdot \frac{1}{1-(1-\frac{1}{2N})^{\frac{1}{t_1}}}. \tag{12}$$

Since $t_1 > 1$, it holds that $(1-\frac{1}{2N})^{-\frac{1}{t_1}} \leq 2^{\frac{1}{t_1}} < 2$. It follows from Bernoulli's inequality that $(1-\frac{1}{2N})^{\frac{1}{t_1}} \leq 1-\frac{1}{2Nt_1}$, and thus $\frac{1}{1-(1-\frac{1}{2N})^{\frac{1}{t_1}}} \leq 2Nt_1$. Inequality (12) becomes

$$H_{(\mathcal{G}(t)\times\mathcal{G}(t),Q(t),0)}((i_1,i_2),\Theta) \leq 4Nt_1. \tag{13}$$

Actually, inequality (13) holds for any starting time, any starting node (i_1,i_2). Thus it holds that $M_{(\mathcal{G}(t),P_{\mathrm{AS}}(t))} = H_{(\mathcal{G}(t)\times\mathcal{G}(t),Q(t),)}(\Theta) \leq 4Nt_1$. This completes the proof. ■

C. Convergence analysis of AS

We are now in the position to characterize the convergence properties of AS. The quantities $T_{\mathrm{con}}(x(0))$ and T_Ψ for AS are defined in a similar way to those in Section III.

Theorem 4.1: Let $x(0) \in \mathbb{R}^N$ and suppose $x(0) \notin \mathcal{W}(x(0))$. Assume that $\mathcal{G}(t)$ be undirected and satisfies Assumption 4.1. Under AS, almost any evolution $x(t)$ starting from $x(0)$ reaches quantized average consensus. Furthermore, $\mathbb{E}[T_{\mathrm{con}}(x(0))] \leq \frac{1}{2}BJ(x(0))^2N^2(8N^6\log(\sqrt{2}N)+1)$.

Proof: The proof is analogous to Theorem 3.1 by using the fact that $\mathbb{E}[T_\Psi] = M_{(\mathcal{G}(t),P_{\mathrm{AS}}(t))}$. ■

V. DISCUSSION

Consider a fixed graph L_N^m with N vertices consists of a clique on m vertices, including vertex i, and a path of length $N-m$ with one end connected to one vertex $k \neq i$ of the clique, and the other end of the path being j. It was shown in [5] that $H_{(L_N^{m_0},P_{\mathrm{SF}})}$ is $O(N^3)$ where $m_0 = \lfloor\frac{2N+1}{3}\rfloor$. Let us consider the case that the algorithm AF is implemented on the graph $L_N^{m_0}$ and initial states $x_i(0) = 0$, $x_j(0) = 2$ and $x_k(0) = 1$ for all $k \neq i,j$. Observe that $\mathbb{E}[T_{\mathrm{con}}(x(0))] = M_{(L_N^{m_0},P_{\mathrm{AF}})}$. From Proposition 3.1, we have that $\mathbb{E}[T_{\mathrm{con}}(x(0))]$ is $O(N^4)$, that is one order less than the bound in Theorem 3.1.

Consider switching graphs $\mathcal{G}(t)$ where $\mathcal{G}(t)$ is the graph $L_N^{m_0}$ defined above when t is a multiple of B; otherwise, all the vertices in $\mathcal{G}(t)$ are isolated. Random walks on $\mathcal{G}(t)$ can be viewed as time-scaled versions of those on $L_N^{m_0}$, that is, random walks on $\mathcal{G}(t)$ only make the movements when t is a multiple of B. Let us consider the case that the algorithm AS is implemented on the graph $L_N^{m_0}$ and initial states $x_i(0) = 0$, $x_j(0) = 2$ and $x_k(0) = 1$ for all $k \neq i, j$. Following the same lines above, we have that the bound on $\mathbb{E}[T_{\mathrm{con}}(x(0))]$ is $O(BN^4)$ which is $N^4 \log N$-order less than that in Theorem 3.1. One of our future work is find a tighter upper bound on $\mathcal{G}(t)$.

REFERENCES

[1] T.C. Aysal, M.J. Coates and M. Rabbat, *Distributed average consensus using probabilistic quantization*, Proceedings of IEEE Workshop on Statistical Signal Processing, Madison, USA, pp. 640 - 644, August 2007.

[2] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and distributed computation: numerical methods*, Prentice Hall, 1989.

[3] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis, *Convergence in multiagent coordination, consensus, and flocking*, Proceedings of the Joint 44th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC'05), Seville, Spain, pp. 2996 - 3000, December 2005.

[4] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, *Randomized gossip algorithms*, IEEE Transactions on Information Theory, Special issue of IEEE Transactions on Information Theory and IEEE ACM Transactions on Networking, vol. 52, No. 6, pp. 2508 - 2530, June 2006.

[5] G. Brightwell and P. Winkler, *Maximum hitting time for random walks on graphs*, Random Structures and Algorithms, vol. 1, No. 3, pp. 263 - 276, 1990.

[6] R. Carli, F. Bullo, and S. Zampieri, *Quantized average consensus via dynamic coding/decoding schemes*, Procedings of IEEE Conference on Decision and Control, Cancun, Mexico, December 2008, To appear.

[7] R. Carli, F. Fagnani, P. Frasca, T. Taylor and S. Zampieri, *Communication constraints in the state agreement problem*, Automatica, to appear.

[8] D. Coppersmith, P. Tetali and P. Winkler, *Collisions among random walks on a graph*, SIAM Journal on Discrete Mathematics, vol. 6, No. 3, pp. 363 - 374, 1993.

[9] Frasca, Carli, Fagnani and Zampieri, *Average consensus by gossip algorithms with quantized communication*, Proceedings of 46th IEEE Conference on Decision and Control (CDC'08), Cancun, Mexico, pp. 4821 - 4836, December 2008.

[10] R.A. Horn and C.R. Johnson, *Matrix analysis*, Cambridge, U.K.: Cambridge University Press, 1987.

[11] A. Jadbabaie, J. Lin, and A. S. Morse, *Coordination of groups of mobile autonomous agents using nearest neighbor rules*, IEEE Transactions on Automatic Control, vol. 48, No. 6, pp. 988 - 1001, June 2003.

[12] A. Kashyap, T. Basar and R. Srikant, *Quantized consensus*, Automatica, vol. 43, pp. 1192 - 1203, July 2007.

[13] Lavaei and Murray, *On quantized consensus by means of gossip algorithm*, Proceedings of the 2009 American Control Conference (ACC'09), St. Louis, USA, to appear.

[14] S. Martínez, *http://faemino.ucsd.edu/soniamartinez/papers/index.html.*

[15] A. Nedich, A. Olshevsky, A. Ozdaglar and J. N. Tsitsiklis, *On distributed averaging algorithms and quantization effects*, MIT LIDS Report 2778, November 2007, https://netfiles.uiuc.edu/angelia/www/nedich.html.

[16] R. Olfati-Saber, J. A. Fax, and R. M. Murray, *Consensus and cooperation in networked multi-Agent systems*, Proceedings of the IEEE, vol. 95, No. 1, pp. 215-233, January 2007.

[17] A. Olshevsky and J.N. Tsitsiklis, *Convergence speed in distributed consensus and averaging*, SIAM Journal on Control and Optimization, to appear.

[18] M. Zhu and S. Martínez, *On the convergence time of distributed quantized averaging algorithms*, Proceedings of the 47th IEEE Conference on Decision and Control, Cancún, Mexico, to appear 2008