On synchronous robotic networks
Part II: Time complexity of rendezvous and deployment algorithms

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***Abstract*— This paper analyzes a number of basic coordination algorithms running on synchronous robotic networks. We provide upper and lower bounds on the time complexity of the move-toward average and circumcenter laws, both achieving rendezvous, and of the centroid law, achieving deployment over a region of interest. The results are derived via novel analysis methods, including a set of results on the convergence rates of linear dynamical systems defined by tridiagonal Toeplitz and circulant matrices.**

I. Introduction

Problem motivation: Although recent years have witnessed the emergence of numerous coordination algorithms for networked mobile systems, the fundamental limits in terms of achievable performance, energy consumption and operational time remain largely unknown. This is partially explained by the inherent difficulty in integrating the various sensing, computing and communication aspects of problems involving groups of mobile agents. In this paper, we consider the problem of analyzing the performance of several coordination algorithms achieving rendezvous and deployment. To this goal, we rely on the general framework proposed in the companion paper [2] to formally model the behavior of robotic networks. Our research effort aims at developing tools and results to assess to what extent coordination algorithms are scalable, and implementable in large networks of mobile agents. Ultimately, we would like to characterize the minimum amount of communication, sensing and control that is necessary to reliably perform a desired task, and we would like to design algorithms that achieve those limits.

Literature review: A survey on cooperative mobile robotics is presented in [3] and an overview of control and communication issues is contained in [4]. Specific topics related to the present treatment include rendezvous [5], [6], [7], [8], [9], cyclic pursuit [10], deployment [11], [12], flocking [13] and consensus [14], [15]. The papers [16], [17], [18] discuss convergence rates of various motion coordination algorithms. See the aforementioned works for references on additional cooperative strategies designed to perform other spatially-distributed tasks.

The complete version of this work is [1]. This paper is submitted to the 2005 CDC jointly with [2].

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Statement of contributions: The companion paper [2] proposes a general framework to model robotic networks and formally analyze their behavior. In particular, this work defines notions of time and communication complexity aimed at capturing the performance and cost of the execution of motion coordination algorithms. Building on these notions, here we establish complexity estimates for various basic motion coordination algorithms that achieve rendezvous and deployment. First, we analyze a simple averaging law for a network of locally-connected agents moving on a line. This law is related to the widely known Vicsek's model, see [13], [19]. We show that this law achieves rendezvous (without preserving connectivity) and that its time complexity belongs to $\Omega(N)$ and $O(N^5)$. Second, for a network of locally-connected agents moving on a line or on a segment, we show that the well-known circumcenter algorithm by [5] has time complexity of order $\Theta(N)$. (This algorithm achieves rendezvous while preserving connectivity with a communication graph with $O(N^2)$ links.) We then consider a network based on a different communication graph, called the limited Delaunay graph, that arises naturally in computational geometry and in the study of wireless communication topologies. For this less dense graph with $O(N)$ communication links, we show that the time complexity of the circumcenter algorithm grows to $\Theta(N^2 \log N)$. For a network of agents moving on $\mathbb{R}^d$ (with a certain communication graph) we introduce a novel "parallel-circumcenter algorithm" and establish its time complexity of order $\Theta(N)$. Third and last, for a network of agents in a one-dimensional environment, we show that the time complexity of the deployment algorithm introduced in [11] is $O(N^3 \log N)$. To obtain these complexity estimates, we develop some novel analysis methods. In particular, we develop a key set of results on linear dynamical systems defined by tridiagonal Toeplitz and circulant matrices that characterize their convergence rates as a function of the matrices dimensions. The interested reader is referred to [1] for a complete discussion of the proofs of all results presented in this manuscript.

Organization: Section II develops some key facts about convergence rates of discrete-time dynamical systems defined by tridiagonal Toeplitz and circulant matrices. These are later employed in obtaining complexity estimates for algorithms achieving rendezvous and deployment. Section III briefly reviews the general approach to the modeling of robotic networks proposed in [2], presenting the notions of control and communication law, coordination tasks and time complexity. Sections IV and V define the rendezvous and deployment coordination tasks, respectively, and present various coordination algorithms that achieve them. For both problems, we establish the asymptotic correctness of the proposed algorithms, and characterize their time complexity. Finally, we present our conclusions in Section VI. In the appendix, we review some basic computational geometric structures employed along the discussion.

Notation: We let `BooleSet` be the set $\{\texttt{true}, \texttt{false}\}$. We let $\prod_{i\in\{1,\dots,N\}} S_i$ denote the Cartesian product of sets $S_1,\dots,S_N$. We let $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ denote the set of strictly positive and non-negative real numbers, respectively. The set of of positive natural numbers is denoted by $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of non-negative integers. If S is a set, then $\operatorname{diag}(S\times S) = \{(s,s)\in S\times S \mid s\in S\}$. For $x\in\mathbb{R}$, we let $\lfloor x\rfloor$ denote the floor of x. For $x\in\mathbb{R}^d$, we denote by $\|x\|_2$ and $\|x\|_\infty$ the Euclidean and the ∞-norm of x, respectively. Recall that $\|x\|_\infty \le \|x\|_2 \le \sqrt{d}\|x\|_\infty$ for all $x\in\mathbb{R}^d$. For $x\in\mathbb{R}^d$ and $r\in\mathbb{R}_+$, we let $B(x,r)$ and $\overline{B}(x,r)$ denote the open and closed ball in $\mathbb{R}^d$ centered at x of radius r, respectively. We let $\mathbf{e}_1,\dots,\mathbf{e}_d$ be the standard orthonormal basis of $\mathbb{R}^d$. Also, we define the vectors $\mathbf{0} = (0,\dots,0)^T$ and $\mathbf{1} = (1,\dots,1)^T$ in $\mathbb{R}^d$. For $f,g\colon\mathbb{N}\to\mathbb{R}$, we say that $f\in O(g)$ (respectively, $f\in\Omega(g)$) if there exist $N_0\in\mathbb{N}$ and $k\in\mathbb{R}_+$ such that $|f(N)|\le k|g(N)|$ for all $N\ge N_0$ (respectively, $|f(N)|\ge k|g(N)|$ for all $N\ge N_0$). If $f\in O(g)$ and $f\in\Omega(g)$, then we use the notation $f\in\Theta(g)$. Finally, we refer the reader to Appendix I for some useful geometric concepts.

II. Tridiagonal Toeplitz and circulant dynamical systems

This section reviews some basic facts about certain classes of Toeplitz matrices, see [20], and other general results that we later employ. For $N\ge 2$ and $a,b,c\in\mathbb{R}$, define the $N\times N$ Toeplitz matrices $\operatorname{Trid}_N(a,b,c)$ and $\operatorname{Circ}_N(a,b,c)$ by

$$\operatorname{Trid}_N(a,b,c) = \begin{bmatrix} b & c & 0 & \dots & 0 \\ a & b & c & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & a & b & c \\ 0 & \dots & 0 & a & b \end{bmatrix},$$

and

$$\operatorname{Circ}_N(a,b,c) = \operatorname{Trid}_N(a,b,c) + \begin{bmatrix} 0 & \dots & \dots & 0 & a \\ 0 & \dots & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ c & 0 & \dots & 0 & 0 \end{bmatrix}.$$

The matrices Trid_N and Circ_N are tridiagonal and circulant, respectively. The two matrices only differ in their $(1,N)$ and $(N,1)$ entries. Note our convention that $C_2(a,b,c) = \begin{bmatrix} b & a+c \\ a+c & b\end{bmatrix}$.

Lemma II.1 (Eigenvalues of tridiagonal Toeplitz and circulant matrices) *For $N\ge 2$ and $a,b,c\in\mathbb{R}$, the following statements hold:*

(i) *for $ac\neq 0$, the eigenvalues and eigenvectors of $\operatorname{Trid}_N(a,b,c)$ are, for $i\in\{1,\dots,N\}$,*

$$b + 2c\sqrt{\frac{a}{c}}\cos\left(\frac{i\pi}{N+1}\right), \quad \begin{bmatrix} \left(\frac{a}{c}\right)^{1/2}\sin\left(\frac{i\pi}{N+1}\right) \\ \left(\frac{a}{c}\right)^{2/2}\sin\left(\frac{2i\pi}{N+1}\right) \\ \vdots \\ \left(\frac{a}{c}\right)^{N/2}\sin\left(\frac{Ni\pi}{N+1}\right) \end{bmatrix};$$

(ii) *the eigenvalues and eigenvectors of $\operatorname{Circ}_N(a,b,c)$ are, for $\omega = \exp(\frac{2\pi\sqrt{-1}}{N})$ and for $i\in\{1,\dots,N\}$,*

$$b + (a+c)\cos\left(\frac{i2\pi}{N}\right) + \sqrt{-1}(c-a)\sin\left(\frac{i2\pi}{N}\right),$$

and $(1,\omega^i,\dots,\omega^{(N-1)i})^T$. •

Remarks II.2 (i) The set of eigenvalues of $\operatorname{Trid}_N(a,b,c)$ is contained in the real interval $[b-2\sqrt{ac}, b+2\sqrt{ac}]$, if $ac\ge 0$, and in the interval in the complex plane $[b-2\sqrt{-1}\sqrt{|ac|}, b+2\sqrt{-1}\sqrt{|ac|}]$, if $ac\le 0$.

(ii) The set of eigenvalues of $\operatorname{Circ}_N(a,b,c)$ is contained in the ellipse on the complex plane with center b, horizontal axis $2|a+c|$ and vertical axis $2|c-a|$.

(iii) Recall from [20] that (1) a square matrix is normal if it has a complete orthonormal set of eigenvectors, (2) circulant matrices and real-symmetric matrices are normal, and (3) if a normal matrix has eigenvalues $\{\lambda_1,\dots,\lambda_n\}$, then its singular values are $\{|\lambda_1|,\dots,|\lambda_n|\}$. •

We can now state the main result of this section.

Theorem II.3 (Tridiagonal Toeplitz and circulant dynamical systems) *Let $N\ge 2$, $\varepsilon\in]0,1[$, and $a,b,c\in\mathbb{R}$. Let $x\colon\mathbb{N}_0\to\mathbb{R}^N$ and $y\colon\mathbb{N}_0\to\mathbb{R}^N$ be solutions to*

$$x(\ell+1) = \operatorname{Trid}_N(a,b,c)\,x(\ell),$$

and

$$y(\ell+1) = \operatorname{Circ}_N(a,b,c)\,y(\ell),$$

with initial conditions $x(0)=x_0$ and $y(0)=y_0$, respectively. The following statements hold:

(i) *if $a=c\neq 0$ and $|b|+2|a|=1$, then $\lim_{\ell\to+\infty} x(\ell) = \mathbf{0}$, and the maximum time required for $\|x(\ell)\|_2 \le \varepsilon\|x_0\|_2$ (over all initial conditions $x_0\in\mathbb{R}^N$) is $\Theta\big(N^2\log\varepsilon^{-1}\big)$;*

(ii) *if $a\neq 0$, $c=0$ and $0<|b|<1$, then $\lim_{\ell\to+\infty} x(\ell) = \mathbf{0}$, and the maximum time required for $\|x(\ell)\|_2 \le \varepsilon\|x_0\|_2$ (over all initial conditions $x_0\in\mathbb{R}^N$) is $O\big(N\log N + \log\varepsilon^{-1}\big)$;*

(iii) *if $a\ge 0$, $c\ge 0$, $b>0$, and $a+b+c=1$, then $\lim_{\ell\to+\infty} y(\ell) = y_{\text{ave}}\mathbf{1}$, where $y_{\text{ave}} = \frac{1}{N}\mathbf{1}^T y_0$, and the maximum time required for $\|y(\ell) - y_{\text{ave}}\mathbf{1}\|_2 \le$*

$\varepsilon\|y_0 - y_{\text{ave}}\mathbf{1}\|_2$ *(over all initial conditions* $y_0 \in \mathbb{R}^N$*) is* $\Theta\big(N^2 \log \varepsilon^{-1}\big)$. •

Next, we extend these results to another interesting set of matrices. For $N \geq 2$ and $a, b \in \mathbb{R}$, define the $N \times N$ matrices $\text{ATrid}_N^+(a, b)$ and $\text{ATrid}_N^-(a, b)$ by

$$\text{ATrid}_N^{\pm}(a,b) = \text{Trid}_N(a,b,a) \pm \begin{bmatrix} a & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & 0 & a \end{bmatrix}.$$

If we define

$$P_+ = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 1 & 0 & \dots & 0 \\ 1 & 0 & -1 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 1 & 0 & \dots & 0 & -1 & 1 \\ 1 & 0 & \dots & 0 & 0 & -1 \end{bmatrix}$$

and

$$P_- = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ (-1)^{N-2} & 0 & \dots & 0 & 1 & 1 \\ (-1)^{N-1} & 0 & \dots & 0 & 0 & 1 \end{bmatrix},$$

then the following similarity transforms are satisfied:

$$\begin{aligned} \text{ATrid}_N^+(a,b) &= P_+ \begin{bmatrix} b+2a & 0 \\ 0 & \text{Trid}_{N-1}(a,b,a) \end{bmatrix} P_+^{-1}, \\ \text{ATrid}_N^-(a,b) &= P_- \begin{bmatrix} b-2a & 0 \\ 0 & \text{Trid}_{N-1}(a,b,a) \end{bmatrix} P_-^{-1}. \end{aligned} \quad (1)$$

To analyze the convergence properties of the dynamical systems determined by $\text{ATrid}_N^+(a,b)$ and $\text{ATrid}_N^-(a,b)$, we recall that $\mathbf{1}^T = (1, \dots, 1) \in \mathbb{R}^N$, and we define $\mathbf{1}_- = (1, -1, 1, \dots, (-1)^{N-2}, (-1)^{N-1})^T \in \mathbb{R}^N$.

Theorem II.4 *Let* $N \geq 2$, $\varepsilon \in]0,1[$, *and* $a, b \in \mathbb{R}$ *with* $a \neq 0$ *and* $|b| + 2|a| = 1$. *Let* $x\colon \mathbb{N}_0 \to \mathbb{R}^N$ *and* $z\colon \mathbb{N}_0 \to \mathbb{R}^N$ *be solutions to*

$$x(\ell+1) = \text{ATrid}_N^+(a,b)\, x(\ell),$$

and

$$z(\ell+1) = \text{ATrid}_N^-(a,b)\, z(\ell),$$

with initial conditions $x(0) = x_0$ *and* $z(0) = z_0$*, respectively. The following statements hold:*

(i) $\lim_{\ell \to +\infty} \big(x(\ell) - x_{\text{ave}}(\ell)\mathbf{1}\big) = \mathbf{0}$, *where* $x_{\text{ave}}(\ell) = (\frac{1}{N}\mathbf{1}^T x_0)(b+2a)^\ell$, *and the maximum time required for* $\|x(\ell) - x_{\text{ave}}(\ell)\mathbf{1}\|_2 \leq \varepsilon\|x_0 - x_{\text{ave}}(0)\mathbf{1}\|_2$ *(over all initial conditions* $x_0 \in \mathbb{R}^N$*) is* $\Theta\big(N^2 \log \varepsilon^{-1}\big)$*;*

(ii) $\lim_{\ell \to +\infty} \big(z(\ell) - z_{\text{ave}}(\ell)\mathbf{1}_-\big) = \mathbf{0}$, *where* $z_{\text{ave}}(\ell) = (\frac{1}{N}\mathbf{1}_-^T z_0)(b-2a)^\ell$, *and the maximum time required for* $\|z(\ell) - z_{\text{ave}}(\ell)\mathbf{1}_-\|_2 \leq \varepsilon\|z_0 - z_{\text{ave}}(0)\mathbf{1}_-\|_2$ *(over all initial conditions* $z_0 \in \mathbb{R}^N$*) is* $\Theta\big(N^2 \log \varepsilon^{-1}\big)$. •

III. SYNCHRONOUS ROBOTIC NETWORKS

The companion paper [2] proposes a formal model for robotic networks, and defines notions of control and communication laws, coordination tasks, and time and communication complexity. For the sake of completeness, we present here simplified versions of these notions.

Definition III.1 *A* uniform network of robotic agents *(or* robotic network*)* $\mathcal{S}$ *is a tuple* $(I, \mathcal{A}, E_{\text{cmm}})$ *consisting of*

(i) $I = \{1, \dots, N\}$*;* I *is called the* set of unique identifiers (UIDs)*;*

(ii) $\mathcal{A} = \{A^{[i]}\}_{i \in I}$*, with* $A^{[i]} = (X, U, X_0, f)$*, is a set of identical control systems; this set is called the* set of physical agents*;*

(iii) E_{cmm} *is a map from* $\prod_{i \in I} X$ *to the subsets of* $I \times I \setminus \text{diag}(I \times I)$*; this map is called the* communication edge map. •

Definition III.2 *A* (synchronous, static, uniform, feedback) control and communication law $\mathcal{CC}$ *for* $\mathcal{S}$ *consists of the sets:*

(i) $\mathbb{T} = \{t_\ell\}_{\ell \in \mathbb{N}_0} \subset \overline{\mathbb{R}}_+$ *is an increasing sequence of time instants, called* communication schedule*;*

(ii) L *is a set containing the* `null` *element, called the* communication language*; elements of* L *are called* messages*;*

and of the maps:

(i) $\text{msg}\colon \mathbb{T} \times X \times I \to L$ *is called* message-generation function*;*

(ii) $\text{ctl}\colon \overline{\mathbb{R}}_+ \times X \times X \times L^N \to U$, $i \in I$*, is called* control function. •

A control and communication law $\mathcal{CC}$ is said to be *time-independent* if the message-generation and control functions are of the form $\text{msg}\colon X \times I \to L$ and $\text{ctl}\colon X \times X \times L^N \to U$, respectively.

Definition III.3 *The* evolution *of* $(\mathcal{S}, \mathcal{CC})$ *from initial conditions* $x_0^{[i]} \in X_0{}^{[i]}$, $i \in I$*, is the set of curves* $x^{[i],\ell}\colon [t_\ell, t_{\ell+1}] \to X$, $i \in I$, $\ell \in \mathbb{N}_0$*, and* $w^{[i]}\colon \mathbb{T} \to W$, $i \in I$*, satisfying*

$$\dot{x}^{[i],\ell}(t) = f\big(x^{[i],\ell}(t),\, \text{ctl}(t, x^{[i],\ell}(t), x^{[i],\ell}(t_\ell), y^{[i]}(t_\ell))\big),$$

where, for $\ell \in \mathbb{N}_0$*, and* $i \in I$*,*

$$x^{[i],\ell}(t_\ell) = x^{[i],\ell-1}(t_\ell)\,,$$

with the convention $x^{[i],-1}(t_0) = x_0^{[i]}$*. Here, the function* $y^{[i]}\colon \mathbb{T} \to L^N$ *(describing the messages received by agent* i*) has components* $y_j^{[i]}(t_\ell)$*, for* $j \in I$*, given by*

$$y_j^{[i]}(t_\ell) = \text{msg}(t_\ell, x^{[j],\ell-1}(t_\ell), i)$$

if $(i,j) \in E_{\text{cmm}}(x^{[1],\ell-1}(t_\ell), \dots, x^{[N],\ell-1}(t_\ell))$ *and* $y_j^{[i]}(t_\ell) = \texttt{null}$ *otherwise.* •

Remarks III.4 (Related concepts and notations) To distinguish between the `null` and the non-`null` messages received by an agent at a given time instant, it is convenient to define the *natural projection* $\pi_L\colon L^N \to 2^L$ that maps an array of messages y to the subset of L containing only the non-`null` messages in y.

In many uniform control and communication laws, the messages interchanged among the network agents are (quantized representations of) the agents' states. The corresponding communication language is $L = X$ and message generation function $\mathrm{msg}_{\mathrm{std}}\colon \mathbb{T} \times X \times I \to X$ is referred to as the *standard message-generation function* and is defined by $\mathrm{msg}_{\mathrm{std}}(t, x, j) = x$. •

Let us now introduce some useful examples of robotic networks. We start with a fairly common example and define some interesting variations.

Example III.5 (Locally-connected first-order agents in $\mathbb{R}^d$) Consider N points $x^{[1]}, \dots, x^{[N]}$ in the Euclidean space $\mathbb{R}^d$, $d \geq 1$, obeying a first-order dynamics $\dot{x}^{[i]}(t) = u^{[i]}(t)$. These are identical agents of the form $A = (\mathbb{R}^d, \mathbb{R}^d, \mathbb{R}^d, (\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_d))$. Assume that each agent can communicate to any other agent within Euclidean distance r, that is, adopt as communication edge map the r-disk proximity edge map $E_{r\text{-disk}}$ defined in Appendix I. These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}} = (I, \mathcal{A}, E_{r\text{-disk}})$. •

Example III.6 (LD-connected first-order agents in $\mathbb{R}^d$) Consider the set of physical agents defined in the previous example. For $r \in \mathbb{R}_+$, recall from Appendix I the r-limited Delaunay map $E_{r\text{-LD}}$ defined by

$$(i,j) \in E_{r\text{-LD}}(x^{[1]}, \dots, x^{[N]}) \quad \text{if and only if} \\ \big(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})\big) \cap \big(V^{[j]} \cap \overline{B}(x^{[j]}, \tfrac{r}{2})\big) \neq \emptyset, \; i \neq j,$$

where $\{V^{[1]}, \dots, V^{[N]}\}$ is the Voronoi partition of $\mathbb{R}^d$ generated by $\{x^{[1]}, \dots, x^{[N]}\}$. These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}} = (I, \mathcal{A}, E_{r\text{-LD}})$. •

Example III.7 (Locally-∞-connected first-order agents in $\mathbb{R}^d$) Consider the set of physical agents defined in the previous two examples. For $r \in \mathbb{R}_+$, define the proximity edge map $E_{r\text{-}\infty\text{-disk}}$ by

$$(i,j) \in E_{r\text{-}\infty\text{-disk}}(x^{[1]}, \dots, x^{[N]}) \quad \text{if and only if} \\ \|x^{[i]} - x^{[j]}\|_\infty \leq r, \; i \neq j.$$

These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-}\infty\text{-disk}} = (I, \mathcal{A}, E_{r\text{-}\infty\text{-disk}})$. •

In order to analyze the performance of a communication and control law, we first define the notion of coordination task, and of task achievement by a robotic network.

Definition III.8 (Coordination task) *Let $\mathcal{S}$ be a robotic network and let $\mathcal{W}$ be a set. A* (static) coordination task *for $\mathcal{S}$ is a map $\mathcal{T}$: $\prod_{i\in I} X^{[i]} \to$ `BooleSet`. Additionally, let $\mathcal{CC}$ a control and communication law for $\mathcal{S}$. The law $\mathcal{CC}$* achieves *the task $\mathcal{T}$ if for all initial conditions $x_0^{[i]} \in X_0^{[i]}$, $i \in I$, the corresponding network evolution $t \mapsto x(t)$ has the property that there exists $T \in \mathbb{R}_+$ such that $\mathcal{T}(x(t)) =$ `true` for all $t \geq T$.* •

The notions of time and communication complexity describe the performance and cost of a control and communication law completing a certain coordination task. Here, we focus on time complexity.

Definition III.9 (Time complexity) *Let $\mathcal{S}$ be a robotic network, let $\mathcal{T}$ be a coordination task for $\mathcal{S}$ and let $\mathcal{CC}$ be a control and communication law for $\mathcal{S}$.*

(i) *The* time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ from $x_0 \in \prod_{i\in I} X_0^{[i]}$ *is*

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0) = \inf\{\ell \mid \\ \mathcal{T}(x(t_k)) = \texttt{true}\,, \textit{ for all } k \geq \ell\}\,,$$

where $t \mapsto (x(t))$ is the evolution of $(\mathcal{S}, \mathcal{CC})$ from x_0.

(ii) *The* time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ *is*

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}) = \sup\Big\{ \mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0) \mid x_0 \in \prod_{i\in I} X_0^{[i]} \Big\}.$$

(iii) *The* time complexity of $\mathcal{T}$ *is*

$$\mathrm{TC}(\mathcal{T}) = \inf\{\mathrm{TC}(\mathcal{T}, \mathcal{CC}) \mid \mathcal{CC} \textit{ achieves } \mathcal{T}\}.$$ •

IV. Rendezvous

In this section, we introduce rendezvous coordination tasks and analyze various coordination algorithms that achieve them, providing upper and lower bounds on their time complexity. Along the section, we will consider the networks $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$ and $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$ presented in Example III.5 andSection III.6.

A. Rendezvous tasks

First, let $\mathcal{S} = (I, \mathcal{A}, E_{\mathrm{cmm}})$ be a uniform robotic network. The *(exact) rendezvous task* $\mathcal{T}_{\mathrm{rndzvs}}\colon X^N \to$ `BooleSet` for $\mathcal{S}$ is the static task defined by $\mathcal{T}_{\mathrm{rndzvs}}(x^{[1]}, \dots, x^{[N]}) =$ `true` if and only if

$$x^{[i]} = x^{[j]}, \text{ for all } (i,j) \in E_{\mathrm{cmm}}(x^{[1]}, \dots, x^{[N]}).$$

Second, let $\mathcal{S} = (I, \mathcal{A}, E_{\mathrm{cmm}})$ be a uniform robotic network with agents' state space $X \subset \mathbb{R}^d$. Examples networks of this form are $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$, see Examples III.5 and IV-B, and $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$, see Examples III.6. For $\varepsilon > 0$, the *ε-rendezvous task* $\mathcal{T}_{\varepsilon\text{-rndzvs}}\colon X^N \to$ `BooleSet` for $\mathcal{S}$ is defined by $\mathcal{T}_{\varepsilon\text{-rndzvs}}(x) =$ `true` if and only if

$$\Big\| x^{[i]} - \operatorname{avrg}\big(\{x^{[i]}\} \cup \{x^{[j]} \mid (i,j) \in E_{\mathrm{cmm}}(x)\}\big) \Big\|_2 < \varepsilon,$$

for all $i \in I$. Here $x = (x^{[1]}, \dots, x^{[N]}) \in X^N \subset (\mathbb{R}^d)^N$. In other words, $\mathcal{T}_{\varepsilon\text{-rndzvs}}$ is `true` at $x \in (\mathbb{R}^d)^N$ if, for all $i \in I$, $x^{[i]}$ is at distance less than ε from the average of its own position with the position of its E_{cmm}-neighbors.

B. Rendezvous without connectivity via the move-toward-average control and communication law

From Example III.5, consider the uniform network $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$ of locally-connected first-order agents in $\mathbb{R}^d$. We now define a static, uniform and time-independent law that we refer to as the move-toward-average law and that we denote by $\mathcal{CC}_{\text{avrg}}$. We loosely describe it as follows:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each communication round each agent transmits its position. Between communication rounds, each agent moves towards and reaches the point that is the average of its neighbors' positions; the average point is computed including the agent's own position.

Next, we *formally* define the law as follows. First, we take $\mathbb{T} = \mathbb{N}_0$ and we assume that each agent operates with the standard message-generation function, i.e., we set $L = \mathbb{R}^d$ and $\text{msg}(x,j) = \text{msg}_{\text{std}}(x,j) = x$. Second, we define the control function $\text{ctl}\colon \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\text{ctl}(x, x_{\text{smpld}}, y) = -k_{\text{prop}} \ \text{vers}\big(x - \text{avrg}(y \cup \{x_{\text{smpld}}\})\big),$$

where $k_{\text{prop}} \geq r$, $\text{vers}\colon \mathbb{R}^d \to \mathbb{R}^d$ is defined by $\text{vers}(0) = 0$ and $\text{vers}(v) = v/\|v\|_2$ for $v \neq 0$, and the map avrg computes the average of a finite point set in $\mathbb{R}^d$:

$$\text{avrg}(S) = \frac{1}{\displaystyle\sum_{p\in\pi_{\mathbb{R}}(S)} 1} \sum_{p\in\pi_{\mathbb{R}}(S)} p\,.$$

In summary we set $\mathcal{CC}_{\text{avrg}} = (\mathbb{N}_0, \mathbb{R}^d, \text{msg}_{\text{std}}, \text{ctl})$. An implementation of this control and communication law is shown in Fig. 1 for $d = 1$. Note that, along the evolution, (1) several agents *rendezvous*, i.e., agree upon a common location, and (2) some agents are connected at the simulation's beginning and not connected at the simulation's end. Finally, we remark that this law is related to the Vicsek's model discussed in [13], [19]. •

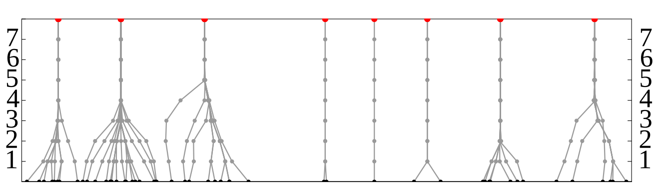

Fig. 1. Evolution of a robotic network under the move-toward-average control and communication law in Example IV-B implemented over the r-disk graph, with $r = 1.5$. The vertical axis corresponds to the elapsed time, and the horizontal axis to the positions of the agents in the real line. The 51 agents are initially randomly deployed over the interval $[-15, 15]$.

The next result characterizes the complexity of this law. The proof can be found in [1].

Theorem IV.1 *For $d = 1$, the network $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$, the law $\mathcal{CC}_{\text{avrg}}$, and the task $\mathcal{T}_{\text{rndzvs}}$ satisfy $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}) \in O(N^5)$ and $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}) \in \Omega(N)$.*

C. Rendezvous with connectivity constraint via the circumcenter control and communication law

Here we define the *circumcenter* control and communication law $\mathcal{CC}_{\text{crcmcntr}}$ for both networks $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$ and $\mathcal{S}_{\mathbb{R}^d,r\text{-LD}}$. This is a uniform, static, time-independent law originally introduced by [5] and later studied in [7], [9]. Loosely speaking, the evolution of the network under the circumcenter control and communication law can be described as follows:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each communication round each agent performs the following tasks: (i) it transmits its position and receives its neighbors' positions; (ii) it computes the circumcenter of the point set comprised of its neighbors and of itself, and (iii) it moves toward this circumcenter while maintaining connectivity with its neighbors.

Let us present this description in more formal terms. We set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\text{msg}^{[i]} = \text{msg}_{\text{std}}$, $i \in I$. In order to define the control function, we need to introduce some preliminary constructions. First, connectivity is maintained by restricting the allowable motion of each agent in the following appropriate manner. If agents i and j are neighbors at time $\ell \in \mathbb{N}_0$, then we require their subsequent positions to belong to $\overline{B}(\frac{x^{[i]}(\ell)+x^{[j]}(\ell)}{2}, \frac{r}{2})$. If an agent i has its neighbors at locations $\{q_1, \dots, q_l\}$ at time ℓ, then its *constraint set* $\mathcal{D}_{x^{[i]}(\ell),r}(\{q_1,\dots,q_l\})$ is

$$\mathcal{D}_{x^{[i]}(\ell),r}(\{q_1,\dots,q_l\}) = \bigcap_{q\in\{q_1,\dots,q_l\}} \overline{B}\Big(\frac{x^{[i]}(\ell)+q}{2}, \frac{r}{2}\Big).$$

Second, in order to maximize the displacement toward the circumcenter of the point set comprised of its neighbors and of itself, each agent solves a convex optimization problem that can be stated in general as follows. For q_0 and q_1 in $\mathbb{R}^d$, and for a convex closed set $Q \subset \mathbb{R}^d$ with $q_0 \in Q$, let $\lambda(q_0, q_1, Q)$ denote the solution to the strictly convex problem:

$$\begin{aligned} &\text{maximize } \lambda \\ &\text{subject to } \lambda \leq 1,\ (1-\lambda)q_0 + \lambda q_1 \in Q. \end{aligned}$$

Under the stated assumptions the solution exists and is unique. Third, note that since the agents operate with the standard message-generation function, it is clear that the natural projection $\pi_{\mathbb{R}^d}$ maps the messages $y^{[i]}(\ell)$ received at time $\ell \in \mathbb{N}_0$ by the agent $i \in I$ onto the positions of its neighbors. We are now ready to define the last constitutive element of $\mathcal{CC}_{\text{crcmcntr}}$. Define the control function $\text{ctl}\colon \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\text{ctl}(x, x_{\text{smpld}}, y) = \lambda_* \cdot (\text{Circum}(\pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}) - x_{\text{smpld}})\,, \tag{2}$$

with $\lambda_* = \lambda(x_{\text{smpld}}, (\text{Circum}(\pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}), \mathcal{D}_{x_{\text{smpld}},r}(\pi_{\mathbb{R}^d}(y)))$. Evolving under this control law, it is clear that, at time $\lfloor t \rfloor + 1$, each agent i reaches the point $(1-\lambda_*)x^{[i]}(\lfloor t \rfloor) + \lambda_* \,\text{Circum}(\pi_{\mathbb{R}^d}(y^{[i]}(\lfloor t \rfloor)) \cup \{x^{[i]}(\lfloor t \rfloor)\})$.

Next, we consider the network $\mathcal{S}_{r\text{-}\infty\text{-disk}}$ of locally-∞-connected first-order agents in $\mathbb{R}^d$, see Example III.7. For this network we define the *parallel circumcenter law*, $\mathcal{CC}_{\text{pll-crcmcntr}}$, by designing d decoupled circumcenter laws running in parallel on each coordinate axis of $\mathbb{R}^d$. As before, this law is uniform, static and time-independent. As before, we set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\text{msg}^{[i]} = \text{msg}_{\text{std}}$, $i \in I$. We define the control function $\text{ctl}\colon \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\text{ctl}(x, x_{\text{smpld}}, y) = \Big(\operatorname{Circum}(\tau_1(\mathcal{M})) - \tau_1(x_{\text{smpld}}), \ldots, \operatorname{Circum}(\tau_d(\mathcal{M})) - \tau_d(x_{\text{smpld}})\Big), \quad (3)$$

where $\mathcal{M} = \pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}$, and $\tau_1, \ldots, \tau_d \colon \mathbb{R}^d \to \mathbb{R}$ denote the canonical projections of $\mathbb{R}^d$ onto $\mathbb{R}$. See Fig. 2 for an illustration of this law in $\mathbb{R}^2$.

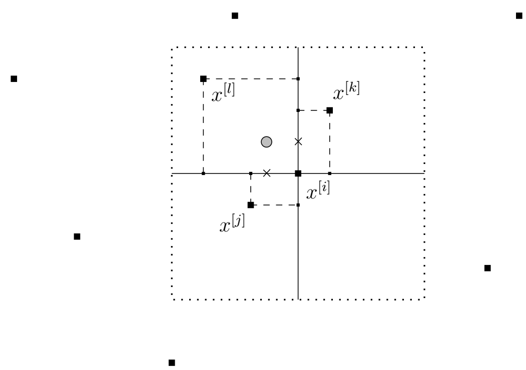

Fig. 2. Parallel circumcenter control and communication law in $\mathbb{R}^2$. The target point for the agent i is plotted in light gray and has coordinates $(\operatorname{Circum}(\tau_1(\mathcal{M}^{[i]})), \operatorname{Circum}(\tau_2(\mathcal{M}^{[i]})))$.

Asymptotic behavior and complexity analysis: The following theorem summarizes the results known in the literature about the asymptotic properties of the circumcenter law.

Theorem IV.2 (Correctness of the circumcenter law) *For $d \in \mathbb{N}$, $r \in \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_+$, the following statements hold:*

(i) *on the network $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$, the law $\mathcal{CC}_{\text{crcmcntr}}$ achieves the exact rendezvous task $\mathcal{T}_{\text{rndzvs}}$;*
(ii) *on the network $\mathcal{S}_{\mathbb{R}^d,r\text{-LD}}$, the law $\mathcal{CC}_{\text{crcmcntr}}$ achieves the ε-rendezvous task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$;*
(iii) *on the network $\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}$, the law $\mathcal{CC}_{\text{pll-crcmcntr}}$ achieves the exact rendezvous task $\mathcal{T}_{\text{rndzvs}}$;*
(iv) *the evolutions of $(\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}, \mathcal{CC}_{\text{crcmcntr}})$, of $(\mathcal{S}_{\mathbb{R}^d,r\text{-LD}}, \mathcal{CC}_{\text{crcmcntr}})$, and of $(\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}, \mathcal{CC}_{\text{pll-crcmcntr}})$ have the property that, if two agents belong to the same connected component of the communication graph at $\ell \in \mathbb{N}_0$, then they continue to belong to the same connected component of the communication graph for all subsequent times $k \geq \ell$.* •

Next we analyze the time complexity of $\mathcal{CC}_{\text{crcmcntr}}$. We provide complete results for the case $d = 1$. As we see next, the complexity of $\mathcal{CC}_{\text{crcmcntr}}$ differs dramatically when applied to the two robotic networks with different communication graphs. The proof of this result relies on Theorems II.3 and II.4 (see [1]).

Theorem IV.3 (Time complexity of circumcenter law) *For $r \in \mathbb{R}_+$ and $\varepsilon \in]0, 1[$, the following statements hold:*

(i) *for $d = 1$, on the network $\mathcal{S}_{\mathbb{R},r\text{-disk}}$, $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(N)$;*
(ii) *for $d = 1$, on the network $\mathcal{S}_{\mathbb{R},r\text{-LD}}$, $\text{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(N^2 \log(N\varepsilon^{-1}))$;*
(iii) *for $d \in \mathbb{N}$, on the network $\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}$, $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{pll-crcmcntr}}) \in \Theta(N)$.* •

Remark IV.4 Theorem IV.3 induces a lower bound on the time communication complexity of the circumcenter law for the higher-dimensional case. Indeed, as a consequence of this result, we have

(i) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{\mathbb{R},r\text{-disk}}$, $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Omega(N)$;
(ii) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{\mathbb{R},r\text{-LD}}$, $\text{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Omega(N^2 \log(N\varepsilon^{-1}))$.

We have performed extensive numerical simulations for the case $d = 2$ and the network $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$. We have ran the algorithm starting from generic initial configurations (where, in particular, agents' positions are not aligned) contained in a bounded region of $\mathbb{R}^2$. We have consistently obtained that the time complexity to achieve $\mathcal{T}_{\text{rndzvs}}$ with $\mathcal{CC}_{\text{crcmcntr}}$ starting from these initial configurations is independent of the number of agents. This leads us to conjecture that, in fact, initial configurations where all agents are aligned (i.e., the 1-dimensional case) give rise to the worst possible performance of the algorithm. In more formal terms, we conjecture that, for $d \geq 2$, $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) = \Theta(N)$. •

V. Deployment

In this section, we introduce the deployment coordination task and analyze a coordination algorithm that achieves it, providing upper and lower bounds on its time complexity. Along the section, we consider the uniform robotic network $\mathcal{S}_{\mathbb{R}^d,r\text{-LD}}$ presented in Example III.6 with parameter $r \in \mathbb{R}_+$. We assume we are given a convex simple polytope $Q \subset \mathbb{R}^d$, with an integrable density function $\phi\colon Q \to \mathbb{R}_+$. We assume that the initial positions of the agents belong to Q and we intend to design a control law that keeps them in Q for subsequent times.

A. Deployment task

By optimal deployment on the convex simple polytope $Q \subset \mathbb{R}^d$ with density function $\phi\colon Q \to \mathbb{R}_+$, we mean the following objective: place the agents on Q so that the expected square Euclidean distance from any point in Q to one of the agents is minimized. To define this task formally, let us review some known preliminary notions; we will require some computational geometric notions from Appendix I. We consider the following network objective function $\mathcal{H}_{\text{deplmnt}}\colon Q^N \to \mathbb{R}$,

$$\mathcal{H}_{\text{deplmnt}}(x^{[1]}, \ldots, x^{[N]}) = \int_Q \min_{i \in I} \|q - x^{[i]}\|_2^2 \, \phi(q) dq\,. \quad (4)$$

This function and variations of it are studied in the facility location and resource allocation research literature; see [21],

[11]. It is convenient [12] to study a generalization of this function. For $r \in \mathbb{R}_+$, define the saturation function $\mathrm{sat}_r : \mathbb{R} \to \mathbb{R}$ by $\mathrm{sat}_r(x) = x$ if $x \leq r$ and $\mathrm{sat}_r(x) = r$ otherwise. For $r \in \mathbb{R}_+$, define the new objective function $\mathcal{H}_{r\text{-deplmnt}} : Q^N \to \mathbb{R}$ by

$$\mathcal{H}_{r\text{-deplmnt}}(x^{[1]}, \dots, x^{[N]}) = \int_Q \min_{i \in I} \mathrm{sat}_{\frac{r}{2}}(\|q - x^{[i]}\|_2^2)\, \phi(q) dq \,. \tag{5}$$

Note that if $r \geq 2\,\mathrm{diam}(Q)$, then $\mathcal{H}_{\text{deplmnt}} = \mathcal{H}_{r\text{-deplmnt}}$. Let $\{V^{[1]}, \dots, V^{[N]}\}$ be the Voronoi partition of Q associated with $\{x^{[1]}, \dots, x^{[N]}\}$. The partial derivative of the cost function takes the following meaningful form

$$\frac{\partial \mathcal{H}_{r\text{-deplmnt}}}{\partial x^{[i]}}(x^{[1]}, \dots, x^{[N]}) = 2\,\mathrm{Mass}(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})) \cdot \\ \cdot \left(\mathrm{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})) - x^{[i]} \right), \quad i \in I \,.$$

(Here, as in Appendix I, $\mathrm{Mass}(S)$ and $\mathrm{Centroid}(S)$ are, respectively, the mass and the centroid of $S \subset \mathbb{R}^d$.) Clearly, the critical points of $\mathcal{H}_{r\text{-deplmnt}}$ are network states where $x^{[i]} = \mathrm{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \frac{r}{2}))$. We call such configurations $\frac{r}{2}$-centroidal Voronoi configurations. For $r \geq 2\,\mathrm{diam}(Q)$, they coincide with the standard centroidal Voronoi configurations on Q. Fig. 3 illustrates these notions.

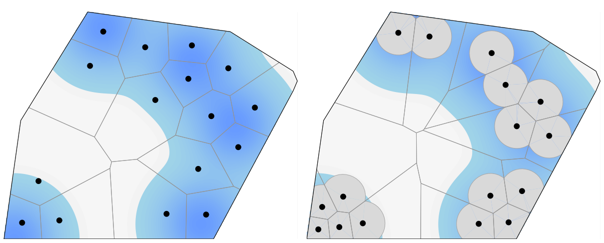

Fig. 3. Centroidal and $\frac{r}{2}$-centroidal Voronoi configurations. The density function ϕ is depicted by a contour plot. For each agent i, the set $V^{[i]} \cap \overline{B}(p_i, \frac{r}{2})$ is plotted in light gray.

Motivated by these observations, we define the following deployment task. For $r, \varepsilon \in \mathbb{R}_+$, define the *$\varepsilon$-$r$-deployment task* $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}} : Q^N \to$ `BooleSet` by $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}(x) =$ `true` if and only if

$$\left\| x^{[i]} - \mathrm{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})) \right\|_2 \leq \varepsilon, \text{ for all } i \in I.$$

Roughly speaking, $\mathcal{T}_{\varepsilon\text{-}r\text{deplmnt}}$ is `true` for those network configurations where each agent is sufficiently close to the centroid of an appropriate region $V^{[i]} \cap \overline{B}(x^{[i]}, \frac{r}{2})$.

B. Centroid law

To achieve the ε-r-deployment task discussed in Example V-A, we define the *centroid* control and communication law $\mathcal{CC}_{\text{centrd}}$. This is a uniform, static, time-independent law studied in [11], [12]. Loosely speaking, the evolution of the network under the centroid control and communication law can be described as follows:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each communication round each agent performs the following tasks: (i) it transmits its position and receives its neighbors' positions; (ii) it computes the centroid of an appropriate region (the region is the intersection between the agent's Voronoi cell and a closed ball centered at its position and of radius $\frac{r}{2}$), and (iii) it moves toward this centroid.

Let us present this description in more formal terms. We set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\mathrm{msg}^{[i]} = \mathrm{msg}_{\text{std}}$, $i \in I$. We define the control function $\mathrm{ctl} : \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\mathrm{ctl}(x, x_{\text{smpld}}, y) = \mathrm{Centroid}(\mathcal{X}) - x_{\text{smpld}} \,,$$

where $\mathcal{X} = Q \cap \overline{B}(x_{\text{smpld}}, \frac{r}{2}) \cap \left(\cap_{p \in \pi_L(y)} H_{x_{\text{smpld}}, p} \right)$ and $H_{x_{\text{smpld}}, p}$ is the half-space $\{q \in \mathbb{R}^d \mid \|q - x_{\text{smpld}}\|_2 \leq \|q - p\|_2\}$. One can show that Q^N is a positively-invariant set for this control law.

The following theorem on the centroid control and communication law summarizes the known results about the asymptotic properties and the novel results on the complexity of this law. In characterizing complexity, we assume $\mathrm{diam}(Q)$ is independent of N, r and ε, and we do not calculate how the bounds depend on r. As for the circumcenter law, we provide complete time-complexity results for the case $d = 1$. The proof of this result relies on Theorems II.3 and II.4 (see [1]).

Theorem V.1 (Time complexity of centroid law) *For $r \in \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_+$, consider the network $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$ with initial conditions in Q. The following statements hold:*

- (i) *for $d \in \mathbb{N}$, the law $\mathcal{CC}_{\text{centrd}}$ achieves the ε-r-deployment task $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}$;*
- (ii) *for $d = 1$ and $\phi = 1$, $\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}, \mathcal{CC}_{\text{centrd}}) \in O(N^3 \log(N\varepsilon^{-1}))$.* •

VI. Conclusions

Building on the framework proposed in the companion paper [2] to model and analyze robotic networks, we have formalized various motion coordination algorithms: the move-toward-average and the circumcenter laws, achieving the rendezvous task, and the centroid law, achieving the deployment task. We have computed the time complexity of these algorithms, providing upper and lower bounds as the number of agents tends to infinity. To obtain these complexity estimates, we have developed some novel analysis methods involving linear dynamical systems defined by tridiagonal Toeplitz and circulant matrices. These results demonstrate the usefulness of the proposed formal model. We hope that they will help assess the complex trade-offs between computation, communication and motion control in robotic networks.

A number of research avenues look now promising and exciting. In this paper, our analysis results essentially consist of a time-complexity analysis of some basic algorithms, but many more open algorithmic questions remain unresolved including (i) analysis of the communication complexity for unidirectional and omnidirectional models of communication; (ii) analysis of other known algorithms for flocking, cohesion, formation, motion planning and a long etcetera; and (iii) complexity analysis results for coordination tasks, as opposed to for algorithms.

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References

[1] S. Martínez, F. Bullo, J. Cortés, and E. Frazzoli, "Synchronous robotic networks and complexity of control and communication laws," *Preprint. Available electronically at* `http://xxx.arxiv.org/math.OC/0501499`, Jan. 2005.

[2] ——, "On synchronous robotic networks – Part I: Models, tasks and complexity notions," in *IEEE Conf. on Decision and Control*, Seville, Spain, Feb. 2005, submitted.

[3] Y. U. Cao, A. S. Fukunaga, and A. Kahng, "Cooperative mobile robotics: Antecedents and directions," *Autonomous Robots*, vol. 4, no. 1, pp. 7–27, 1997.

[4] E. Klavins and R. M. Murray, "Distributed algorithms for cooperative control," *IEEE Pervasive Computing*, vol. 3, no. 1, pp. 56–65, 2004.

[5] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita, "Distributed memoryless point convergence algorithm for mobile robots with limited visibility," *IEEE Transactions on Robotics and Automation*, vol. 15, no. 5, pp. 818–828, 1999.

[6] P. Flocchini, G. Prencipe, N. Santoro, and P. Widmayer, "Gathering of asynchronous oblivious robots with limited visibility," *Theoretical Computer Science*, 2005, to appear.

[7] J. Lin, A. S. Morse, and B. D. O. Anderson, "The multi-agent rendezvous problem - the asynchronous case," in *IEEE Conf. on Decision and Control*, Paradise Island, Bahamas, Dec. 2004, pp. 1926–1931.

[8] Z. Lin, M. Broucke, and B. Francis, "Local control strategies for groups of mobile autonomous agents," *IEEE Transactions on Automatic Control*, vol. 49, no. 4, pp. 622–629, 2004.

[9] J. Cortés, S. Martínez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in d dimensions," *IEEE Transactions on Automatic Control*, June 2004, submitted. Electronic version available at `http://www.me.ucsb.edu/bullo`.

[10] J. Marshall, M. Broucke, and B. Francis, "Formations of vehicles in cyclic pursuit," *IEEE Transactions on Automatic Control*, vol. 49, no. 11, pp. 1963– 1974, 2004.

[11] J. Cortés, S. Martínez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Transactions on Robotics and Automation*, vol. 20, no. 2, pp. 243–255, 2004.

[12] J. Cortés, S. Martínez, and F. Bullo, "Spatially-distributed coverage optimization and control with limited-range interactions," *ESAIM. Control, Optimisation & Calculus of Variations*, Jan. 2004, to appear. Electronic version available at `http://www.me.ucsb.edu/bullo`.

[13] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[14] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[15] W. Ren and R. W. Beard, "Consensus seeking in multi-agent systems using dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, Dec. 2003, to appear.

[16] M. Savchenko and E. Frazzoli, "On the time complexity of the multiple-vehicle coordination problem," in *American Control Conference*, Portland, OR, Sept. 2004, to appear.

[17] Y. Hatano and M. Mesbahi, "Agreement over random networks," in *IEEE Conf. on Decision and Control*, Paradise Island, Bahamas, Dec. 2004, pp. 2010–2015.

[18] S. L. Smith, M. E. Broucke, and B. A. Francis, "A hierarchical cyclic pursuit scheme for vehicle networks," *Automatica*, Jan. 2005, to appear.

[19] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, no. 6-7, pp. 1226–1229, 1995.

[20] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*. Philadelphia, PA: SIAM, 2001.

[21] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu, *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, 2nd ed., ser. Wiley Series in Probability and Statistics. New York, NY: John Wiley & Sons, 2000.

[22] M. de Berg, M. van Kreveld, and M. Overmars, *Computational Geometry: Algorithms and Applications*. New York: Springer Verlag, 1997.

Appendix I
Basic geometric notions

Here we have gathered various geometric concepts used throughout the paper. Let $S \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be compact. The *circumcenter of* S, denoted by $\operatorname{Circum}(S)$, is the center of the smallest-radius sphere in $\mathbb{R}^d$ enclosing S. Given an integrable function $\phi\colon S \to \mathbb{R}_+$, the *mass* of S is $\operatorname{Mass}(S) = \int_S \phi(q)dq$, and the *centroid* of S is

$$\operatorname{Centroid}(S) = \frac{1}{\operatorname{Mass}(S)} \int_S q\phi(q)dq\,.$$

A *partition* of S is a collection of subsets of S with disjoint interiors and whose union is S. Given a set of N distinct points $\mathcal{P} = \{p_i\}_{i\in\{1,\dots,N\}}$ in S, the *Voronoi partition of* S generated by $\mathcal{P}$ (with respect to the Euclidean norm) is the collection of sets $\{V_i(\mathcal{P})\}_{i\in\{1,\dots,N\}}$ defined by $V_i(\mathcal{P}) = \{q \in S \mid \|q - p_i\|_2 \le \|q - p_j\|_2\,,\ \text{for all } p_j \in \mathcal{P}\}$. We usually refer to $V_i(\mathcal{P})$ as V_i. For a detailed treatment of Voronoi partitions we refer to [22], [21].

For $I = \{1,\dots,N\}$ and $S \subset \mathbb{R}^d$, a *proximity edge map* is a map of the form $E\colon S^N \to 2^{I\times I\setminus\operatorname{diag}(I\times I)}$. For $r \in \mathbb{R}_+$, we define the r-disk proximity edge map $E_{r\text{-disk}}\colon (\mathbb{R}^d)^N \to 2^{I\times I}$ and the r-limited Delaunay proximity edge map $E_{r\text{-LD}}\colon (\mathbb{R}^d)^N \to 2^{I\times I}$ as follows. An edge $(i,j) \in I\times I \setminus \operatorname{diag}(I\times I)$ belongs to $E_{r\text{-disk}}(x_1,\dots,x_N)$ if and only if $\|x_i - x_j\|_2 \le r$. An edge $(i,j) \in I\times I\setminus\operatorname{diag}(I\times I)$ belongs to $E_{r\text{-LD}}(x_1,\dots,x_N)$ if and only if

$$\big(V_i \cap \overline{B}(x_i, \tfrac{r}{2})\big) \cap \big(V_j \cap \overline{B}(x_j, \tfrac{r}{2})\big) \neq \emptyset,$$

where $\{V_1,\dots,V_N\}$ is the Voronoi partition of $\mathbb{R}^d$ generated by $\{x_1,\dots,x_N\}$. Illustrations of these concepts are given in Fig. 4.

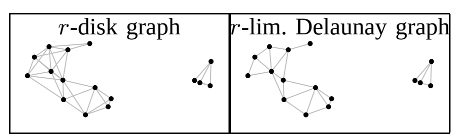

Fig. 4. The r-disk and r-limited Delaunay graphs in $\mathbb{R}^2$

As proved in [12], the r-limited Delaunay graph and the r-disk graph have the same connected components. Additionally, the r-limited Delaunay graph is "computable" on the r-disk graph in the following sense: any node in the network can compute the set of its neighbors in the r-limited Delaunay graph if it is given the set of its neighbors in the r-disk graph. This implies that any control and communication law for a network with communication graph $E_{r\text{-LD}}$ can be implemented on a analogous network with communication graph $E_{r\text{-disk}}$.